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ON RESISTIVE INSTABILITIES

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CONTENTS

It has been established by Furth, Killeen & Rosenbluth (1963), and by Johnson, Greene & Coppi (1963), that a hydromagnetic equilibrium which is stable on a theory in which electrical resistance is ignored, may yet be unstable through finite conductivity effects. These authors have isolated and categorized several types of such instabilities which, they show, originate from the critical layer in which the perturbation wavefront is perpendicular to the equilibrium magnetic field.

In this paper, the asymptotic properties of the critical layer equations, for large values of the critical layer coordinate, are obtained in a number of cases of interest, using the sheet pinch model with uniform resistivity. The mathematical approach is a novel variant of the Laplace integral representation, which allows results of greater generality to be obtained than those given by previous authors. The technique is applied first to the slow interchange mode, and the restricted (but most significant) class of solutions found by Johnson *et al.* is recovered. It is also shown that modes entirely localized within the critical layer do not occur. Such modes do exist for the more rapid interchange modes, and a new discussion of these is presented.

Finally, the oscillatory resistive modes, which arise when the perturbation wavefront is not perpendicular to the equilibrium magnetic field, are studied by a similar mathematical method, and a class of eigenvalues is obtained.

1. Introduction

During the past twenty years, the attempt to design a controlled thermonuclear reactor has led to considerable theoretical effort being expended in categorizing and understanding instabilities which can arise in a plasma. The simplest models treat the plasma as an electrically conducting incompressible fluid with small resistivity r. Of these models, that known as the sheet pinch leads to the simplest analytical description of the stability of the fluid. In the Boussinesq

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approximation, the linear stability problem gives rise to a regular fourth-order ordinary differential equation, which in the ideal limit $r \to 0$ reduces to a singular second-order equation.

The situation is thus very similar to that encountered in the study of the linear stability of plane parallel flows with small viscosity, ν, where the governing equation is the regular fourth-order Orr–Sommerfeld equation, which reduces to the singular second-order Rayleigh equation in the inviscid limit $\nu \to 0$. It is well known in this theory (see, for example, Lin 1955) that velocity profiles, stable according to inviscid theory, may yet be found unstable when the effects of non-zero viscosity are properly allowed for; i.e. the viscous diffusion process may be destabilizing. Analogous results hold for the plasma model.

Using an energy argument, Suydam (1958, see also Roberts 1967, ch. 9) obtained a necessary condition for the linear stability of the ideal ($r = 0$) model. Later Newcomb (1960) found situations in which Suydam's condition was also sufficient. Then, in a notable paper, Furth, Killeen & Rosenbluth (1963), referred to here as F.K.R., recognized that a magnetostatic equilibrium, stable according to ideal theory, may yet be found unstable when the effects of non-zero resistivity are properly allowed for; i.e. the ohmic diffusion process may be destabilizing. The situation is, however, more complex than the hydrodynamic situation, and they found several distinct types of instability; see also Johnson, Greene & Coppi (1963), referred to here as J.G.C. and Coppi, Greene & Johnson (1966).

When r is small but nonzero, the disturbance is governed by the singular second-order equation of ideal theory everywhere, except in a critical layer around the singularity and, in general, at the boundaries. In these regions the full fourth-order resistive equations are required. In particular, it is necessary to determine which linear combination of the four solutions of the resistive equation connects the ideal solutions (assumed matched to the boundaries) on either side of the critical layer. This process has been described both by F.K.R. and J.G.C. in cases where the perturbation wavefront is perpendicular to the equilibrium field in the critical layer. Although these authors made no effort to solve the general matching problem, it appears that they succeeded in locating the cases of prime physical interest. They found that some instabilities, such as the fast interchange modes, are entirely localized within the critical layer, while others, such as the tearing modes, are not.

Much of this paper is concerned with a detailed analysis of the asymptotic structure of the solutions of the critical layer equation (for large values of the critical layer coordinate) in the case of the slow interchange instability. The application of the Laplace method meets unexpected points of difficulty. Their elucidation provides the main mathematical motivation and interest of the present paper. The leading terms of the asymptotic expansions, in the complete sense of Olver (1964), are determined for all solutions in each of the four Stokes sectors which arise. These are used to match, across the critical layer, the ideal solutions of the physical problem. It is then discovered that none of the slow interchange modes are localized within the critical layer. Certain modes, which are identified with those found by J.G.C., grow at a rate which is determined entirely by the critical layer, irrespective of the location and nature of the distant boundaries or of the ideal solutions connecting them to the critical layer. In such cases, the critical layer may be said to be 'active', in contrast to the 'passive' layers (such as those occurring for the tearing modes) which, while they influence the growth rate, do not determine it without reference to the distant boundaries. In principle, the present analysis allows passive layers to be treated as readily as active layers; this advantage is not enjoyed by the analytical methods employed by J.G.C. and F.K.R., the scope of which are 'restricted'.

The basic analysis can also be used in two other ways. First, the propagation of waves in which the perturbation wavefront and the equilibrium magnetic field are not perpendicular may be discussed from a related mathematical standpoint; all waves associated with active critical layers are shown to be stable. Secondly, the localized modes arising at large perturbation wavenumbers may be obtained, and complete solutions can be readily given in all cases except that of the fast interchange mode.

While, perhaps, none of these new results are as significant as those of J.G.C. and F.K.R., they do bring the theory closer to completion, and also demonstrate a mathematical tool of some independent interest.

2. The governing equations

The sheet pinch consists of electrically conducting incompressible fluid, contained between two parallel walls, $z = z_A$ and $z = z_B$ say, in the presence of a sheared magnetic field

$$\boldsymbol{B}_0 = [B_{0x}(z), B_{0y}(z), 0], \tag{2.1}$$

where (x, y, z) are rectangular cartesian coordinates with z-axis vertically upwards. An externally applied uniform gravitational field, $\boldsymbol{g}$, which is necessary for the existence of the interchange modes, acts in the negative z-direction. There is a density stratification:

$$\rho_0 = \rho_c(1+\beta z), \tag{2.2}$$

where ρ_c and β are constants. The quantities $|\beta z_A|$ and $|\beta z_B|$ are both assumed to be so small compared with unity that the Boussinesq approximation may be used. The resistivity, r, is supposed uniform, thus eliminating the rippling modes of F.K.R. If $\boldsymbol{u}$ is the fluid velocity, and $\boldsymbol{b} = \boldsymbol{B} - \boldsymbol{B}_0$ is the perturbation in magnetic field, the linearized induction, fluid motion and continuity equations are, ignoring viscous diffusion,

$$\partial \boldsymbol{b}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) + \eta \nabla^2 \boldsymbol{b}, \tag{2.3}$$

$$\partial \boldsymbol{u}/\partial t = -\nabla \varpi + (\boldsymbol{B}_0 \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{B}_0)/(\mu \rho_c) - g\rho' \hat{\boldsymbol{z}}/\rho_c, \tag{2.4}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0, \tag{2.5}$$

where $\hat{\boldsymbol{z}}$ is a unit vector directed upwards, μ is the (constant) permeability, η $(= r/\mu)$ is the magnetic diffusivity, and $\rho_c \varpi$ denotes the leading order pressure perturbation. Since the fluid is incompressible, the density perturbation, ρ', satisfies in the present theory

$$\partial \rho'/\partial t = -\beta \rho_c u_z. \tag{2.6}$$

Perturbations will be expanded into normal modes of the form

$$\boldsymbol{b} = \boldsymbol{b}(z) \exp(\mathrm{i}\alpha_x x + \mathrm{i}\alpha_y y + \alpha p t), \tag{2.7}$$

where p and $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, 0)$ are constant. Let $\mathscr{B}$ be a typical strength of $\boldsymbol{B}_0$, and introduce

$$F = \boldsymbol{\alpha} \cdot \boldsymbol{B}_0/(|\boldsymbol{\alpha}| \mathscr{B}), \tag{2.8}$$

a quantity which vanishes when the perturbation wavefront is perpendicular to $\boldsymbol{B}_0$. It is found that (2.3) to (2.6) lead to the fourth-order dimensionless system

$$(\mathrm{D}^2 - \alpha^2 - S\alpha p)\,\psi = FW, \tag{2.9}$$

$$(\mathrm{D}^2 - \alpha^2 - S\alpha F^2/p + G/p^2)\,W = -(S\alpha/p)\,(\mathrm{D}^2 F - S\alpha p F)\,\psi. \tag{2.10}$$

Here $$\mathrm{D} = \mathrm{d}/\mathrm{d}z, \quad \psi = \mathrm{i}b_z/\alpha, \quad W = u_z, \quad G = g\beta\tau_a^2, \quad S = \tau_\eta/\tau_a,$$

where the ideal (Alfvénic) time-scale, τ_a, and the resistive time-scale, τ_η, are defined by

$$\tau_a = \mathscr{L}(\mu\rho_c)^{\frac{1}{2}}/\mathscr{B}, \quad \tau_\eta = \mathscr{L}^2/\eta,$$

and $\mathscr{L} = z_{\mathrm{B}} - z_{\mathrm{A}}$. In making these equations non-dimensional, τ_a has been used as unit of time and $\eta/\mathscr{L}$ as unit of velocity. It should be noted that, as $r \to 0$, the Lundquist number, S, tends to infinity. It is sometimes convenient to work in terms of ψ alone; (2.9) and (2.10) give

$$(\mathrm{D}^2-\alpha^2)\,F^{-1}(\mathrm{D}^2-\alpha^2)\,\psi = (S\alpha/p)\,[\mathrm{D}(p^2+F^2)\,\mathrm{D}(F^{-1}\psi) - \alpha^2(p^2+F^2)\,(F^{-1}\psi)] - (G/p^2F)\,(\mathrm{D}^2-\alpha^2-S\alpha p)\,\psi. \quad (2.11)$$

For definiteness, the boundaries will be supposed to be rigid and perfectly conducting, so that

$$\psi = W = 0, \quad \text{for} \quad z = z_{\mathrm{A}} \quad \text{and} \quad z = z_{\mathrm{B}}. \quad (2.12)$$

The interval $z_{\mathrm{A}} \leqslant z \leqslant z_{\mathrm{B}}$ will be denoted by $\mathscr{I}$. The direction, and generally the magnitude also, of $\boldsymbol{a}$ will be assumed given. For a given G, (2.9) to (2.12) then present an eigenvalue problem for p.

If the resistivity is zero, it is found, formally setting $S\alpha/p = \infty$, that (2.9) to (2.11) become

$$\phi = -FW, \quad (2.13)$$

$$(\mathrm{D}^2-\alpha^2+G/p^2)\,W = p^{-2}[F(\mathrm{D}^2-\alpha^2)\,\phi - \phi\mathrm{D}^2F], \quad (2.14)$$

$$\mathrm{D}(p^2+F^2)\,\mathrm{D}W = \alpha^2(p^2+F^2)\,W - GW, \quad (2.15)$$

where $\phi = S\alpha p\psi$. Solutions to these equations are *the ideal solutions*.

There are three main issues to face when considering the limit $S\alpha/p \to \infty$. First, the limit is a singular one: although (2.11) is of the fourth order, (2.15) is only of second order. In general boundary layers must be anticipated at the walls which match the interior flow, determined by (2.13) to (2.15) to leading order, to the boundary conditions (2.12). These layers can be constructed only if the interior flow satisfies

$$W = 0, \quad \text{for} \quad z = z_{\mathrm{A}} \quad \text{and} \quad z = z_{\mathrm{B}}, \quad (2.16)$$

to leading order. Indeed, by (2.13), all conditions (2.12) are then obeyed, and the need for the boundary layer disappears at leading order. In cases more general than (2.12), the boundary layers are present but, since they are passive, they are of no great interest, unless the critical point (see below) lies asymptotically within one of them.

Secondly, the limit introduces singularities: although (2.11) is regular everywhere, (2.15) possesses regular† singularities at the points $z = z_1$ and $z = z_2$ satisfying

$$F(z_r) = (-1)^{r-1}\,\mathrm{i}p \quad (r = 1, 2). \quad (2.17)$$

It is worth noting that, when these points lie in or (in the asymptotic sense given later) close to the interval $\mathscr{I}$, it is necessary to delineate carefully the Stokes and anti-Stokes lines of (2.11) associated with these critical points in order to decide the conditions under which ideal solutions provide uniformly valid solutions to the full equations in the limit $S\alpha/p \to \infty$.

† The singularities are regular only if p^2+F^2 has no more than a quadratic zero at $z = z_1$ and $z = z_2$, as will be supposed here. More than one pair of such zeros may exist, and the present theory must then be modified to include all of them. It is assumed throughout that $\boldsymbol{B}_0$ is continuously varying in z, so that no choice of the direction of $\boldsymbol{a}$ will cause F to vanish in any finite sub-interval of $\mathscr{I}$.

The third issue concerns the question of how completely the eigenvalue spectrum can be covered by the ideal system (2.13) to (2.16). This possesses solutions, called *hydromagnetic modes*, which are the limits, as $S\alpha/p \to \infty$, of solutions to the full system (2.9) to (2.12). It is, however, also possible to find solutions of the full system which do not so reduce. *Resistive modes*, for which $p \to 0$ as $S\alpha/p \to \infty$, are particularly significant, since they can be unstable in situations in which all hydromagnetic modes are stable. For the purposes of the present discussion, we divide them into two classes depending on whether F possesses a zero or not in $\mathscr{I}$; i.e. whether the perturbation wavefront is perpendicular to the magnetic field anywhere in $\mathscr{I}$, or not.

Suppose $F(z_c) = 0$, where z_c is real ($z_A < z < z_B$) and† $F'(z_c) \neq 0$. Since $p \to 0$ as $S\alpha/p \to \infty$, the regular singularities z_1 and z_2 of (2.17) coalesce with z_c in this limit. Writing

$$F \simeq F_c'(z-z_c) \quad \text{as} \quad z \to z_c, \tag{2.18}$$

(2.15) gives

$$\phi \simeq A(z-z_c)^{\frac{1}{2}+2m} + B(z-z_c)^{\frac{1}{2}-2m} \quad \text{as} \quad z \to z_c, \tag{2.19}$$

where A and B are constants, and

$$4m^2 = \tfrac{1}{4} - G/F_c'^2. \tag{2.20}$$

(In cases in which $G/F_c'^2 < \frac{1}{4}$, the positive root of (2.20) will be selected for m.) In the neighbourhood of a critical layer of width δ (where $\delta \to 0$ as $S\alpha/p \to \infty$), the full equations must be used to describe the solutions. Introducing the stretched coordinate, ζ, by

$$z - z_c = \zeta\delta, \tag{2.21}$$

(2.11) gives to leading order

$$\left[\frac{d^2}{d\zeta^2} - \left(\frac{S\alpha F_c'^2\delta^4}{p}\right)\zeta^2 + \left(\frac{G}{p^2} - \alpha^2\right)\delta^2\right]\frac{1}{\zeta}\left[\frac{d^2}{d\zeta^2} - (S\alpha p + \alpha^2)\,\delta^2\right]\psi = -\frac{S\alpha F_c'\delta^5}{p}[D^2F - S\alpha p F_c'\delta\zeta]\,\psi. \tag{2.22}$$

It is this equation which leads to the resistive modes of F.K.R. and J.G.C.

If we now assume that $\alpha \leqslant O(1)$, so that all terms in α^2 can be ignored in (2.22), then the remaining terms on the left-hand side of (2.22) are of the same order provided both $S\alpha\delta^4/p$ and $S\alpha p\delta^2$ are $O(1)$ in the limit, i.e. provided that

$$\delta = O((S\alpha)^{-\frac{1}{3}}) \quad \text{and} \quad p = O((S\alpha)^{-\frac{1}{3}}). \tag{2.23}$$

This defines the slow interchange mode, one of the main topics of this paper. We find it more convenient to replace (2.23) by

$$\delta = (p/S\alpha F_c'^2)^{\frac{1}{4}}, \quad A = (S\alpha p^3/F_c'^2)^{\frac{1}{2}}, \tag{2.24}$$

and to require that A, the new eigenvalue parameter replacing p, is $O(1)$ in the limit; for definiteness, we choose $\arg\delta = \frac{1}{4}\arg p$, where it is supposed that $-\pi < \arg p \leqslant \pi$. Equations (2.9) to (2.11) then become

$$\frac{d^2\Psi}{d\zeta^2} = A(\Psi + \zeta V), \tag{2.25}$$

$$A\frac{d^2V}{d\zeta^2} = (4m^2 - \tfrac{1}{4})\,V + \zeta\frac{d^2\Psi}{d\zeta^2}, \tag{2.26}$$

$$\frac{d}{d\zeta}\left[\frac{1}{\zeta}\frac{d^2\Psi}{d\zeta^2}\right] = \frac{d}{d\zeta}\left[(\zeta^2 + A)\frac{d}{d\zeta}\left(\frac{\Psi}{\zeta}\right)\right] + \frac{4m^2 - \frac{1}{4}}{A\zeta}\left[\frac{d^2}{d\zeta^2} - A\right]\Psi, \tag{2.27}$$

where

$$\Psi = S\alpha A^{\frac{2}{3}}\psi \quad \text{and} \quad V = A^{\frac{1}{6}}W.$$

† See footnote on previous page.

As well as possessing ideal solutions of the form (2.19), equation (2.11) possesses *resistive solutions* which, as the W.K.B.J. method shows, behave as linear combinations of

$$(z-z_c)^{2k-\frac{5}{2}}\exp\left\{\tfrac{1}{2}(z-z_c)^2\left(\frac{S\alpha}{p}\right)^{\frac{1}{2}}F_c'\right\} \quad\text{and}\quad (z-z_c)^{-2k-\frac{1}{2}}\exp\left\{-\tfrac{1}{2}(z-z_c)^2\left(\frac{S\alpha}{p}\right)^{\frac{1}{2}}F_c'\right\}, \tag{2.28}$$

where

$$k=\frac{m^2}{A}+\frac{A}{4}-\frac{1}{16A}+\frac{1}{2}. \tag{2.29}$$

Under the transformation (2.21) they, like (2.19), must be related to solutions of the critical layer equation (2.27) for large $|\zeta|$; see § 4 below. Formal solutions of the critical layer equations (2.25) to (2.27) are derived in § 3, and their asymptotic expansions are obtained in § 4. Certain significant exceptional cases are examined in § 5.

When F does not vanish in $\mathscr{I}$, interest is centred on values of p which are purely imaginary in leading order in the large $S\alpha/p$ limit, and for which one (or both) of the critical points defined by (2.17) lie in $\mathscr{I}$. Boris (1968) has shown that such resistive modes occur only for critical points located (asymptotically as $S\alpha/p\to\infty$) at global extrema of F for $z\in\mathscr{I}$. These points may be located either at the end-points z_A and z_B, or at internal points of $\mathscr{I}$ where F attains a global maximum or minimum. The problem posed by the former situation is analogous to the usual parallel flow problem (cf. Reid 1965), and has been completely solved by Boris. The situation posed by the latter leads to a critical layer equation whose general asymptotic properties raise difficulties which he did not fully resolve.

Assume, therefore, that F has an extremum at $z=z_c$ within $\mathscr{I}$, so that

$$F\simeq F_c+\tfrac{1}{2}F_c''(z-z_c)^2, \quad\text{as}\quad z\to z_c, \tag{2.30}$$

where F_c and F_c'' are non-zero. To find the appropriate critical layer equation, let

$$\delta=(\pm S\alpha F_c''/\mathrm{i})^{-\frac{1}{4}},\quad p=\pm(\mathrm{i}F_c-\tfrac{1}{2}\mathrm{i}AF_c''\delta^2),\quad B=G/F_cF_c'', \tag{2.31}$$

where that value of δ with the smallest value of $|\arg\delta|$ is selected in each case, and A, the new eigenvalue parameter, is $O(1)$. Equation (2.22) then gives

$$\frac{\mathrm{d}^4\psi}{\mathrm{d}\zeta^4}=\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[(\zeta^2+A)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right]+B\psi. \tag{2.32}$$

The properties of this equation as $|\zeta|\to\infty$ are considered in § 7, and the related oscillatory resistive modes are discussed. The investigation is made to depend on the analysis of §§ 3 to 5.

3. Solution of the critical layer equations

Equations (2.25) to (2.27) may be solved by Laplace transformation. Let

$$\left.\begin{aligned}\Psi(\zeta)&=\int_{\mathscr{C}}\mathrm{e}^{s\zeta}\,Q(s)\,\mathrm{d}s,\\ V(\zeta)&=-\frac{1}{A}\int_{\mathscr{C}}\mathrm{e}^{s\zeta}\,K(s)\,\mathrm{d}s,\end{aligned}\right\} \tag{3.1}$$

where $\mathscr{C}$ is a contour in the complex s plane chosen in such a way that the integrated parts, arising in the derivation of (3.2) to (3.4) below, vanish. On substitution of (3.1) into (2.25) to (2.27), we obtain

$$\frac{\mathrm{d}K}{\mathrm{d}s}=(s^2-A)\,Q, \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(As^2Q)=(As^2-4m^2+\tfrac{1}{4})\,K, \tag{3.3}$$

and
$$\frac{d}{ds}\left(\frac{As^2}{A-s^2}\frac{dK}{ds}\right)+(As^2-4m^2+\tfrac{1}{4})K=0. \tag{3.4}$$

These may be solved in terms of the confluent hypergeometric function. Denote by $M(z)$ solutions of
$$\frac{d^2M}{dz^2}+\left(-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)M=0, \tag{3.5}$$
where k is defined by (2.29). It may be verified by direct substitution that
$$Q(s)=s^{-\frac{1}{2}}M'(s^2)+\tfrac{1}{2}s^{-\frac{3}{2}}[1+s^{-2}(A+\tfrac{1}{2}-4k)]\,M(s^2), \tag{3.6}$$
and
$$K(s)=s^{\frac{1}{2}}M'(s^2)+\tfrac{1}{2}s^{\frac{1}{2}}[1-s^{-2}(A+\tfrac{1}{2})]\,M(s^2), \tag{3.7}$$
satisfy (3.2) to (3.4), two independent solutions being, for instance, provided by
$$M(z)=M_{k,\pm m}(z), \tag{3.8}$$
where
$$M_{k,m}(z)=z^{\frac{1}{2}+m}\,e^{-\frac{1}{2}z}\,{}_1F_1(\tfrac{1}{2}+m-k;1+2m;z), \tag{3.9}$$
and ${}_1F_1(a;b;z)$ is Kummer's hypergeometric function. By the use of (2.29), (3.9) and elementary properties of Kummer's function (see, for example, Slater 1960), equations (3.6) and (3.7) may also be written as
$$Q(s)\equiv Q(s,A,k,m)=(1/A)\,(\tfrac{1}{4}+m-\tfrac{1}{2}A)\,e^{-\frac{1}{2}s^2}s^{2m-\frac{3}{2}}$$
$$\times\{(\tfrac{1}{4}-m+\tfrac{1}{2}A)\,{}_1F_1(\tfrac{3}{2}+m-k;1+2m;s^2)+(\tfrac{1}{4}-m-\tfrac{1}{2}A)\,{}_1F_1(\tfrac{1}{2}+m-k;1+2m;s^2)\}, \tag{3.10}$$
and
$$K(s)\equiv K(s,A,k,m)=(1/A)\,(\tfrac{1}{4}+m-\tfrac{1}{2}A)\,e^{-\frac{1}{2}s^2}s^{2m-\frac{1}{2}}$$
$$\times\{(\tfrac{1}{4}-m+\tfrac{1}{2}A)\,{}_1F_1(\tfrac{3}{2}+m-k;1+2m;s^2)-(\tfrac{1}{4}-m-\tfrac{1}{2}A)\,{}_1F_1(\tfrac{1}{2}+m-k;1+2m;s^2)\}, \tag{3.11}$$
a second set of solutions being given by reversing the sign of m. Any linear combination of these two sets is also allowed, and the following choice will be of importance later:
$$Q(s)=Q_r(s)=\frac{e^{\pi irm}}{\Gamma(\frac{1}{2}-m-(-1)^r k)}\,q(m,s)-\frac{e^{-\pi irm}}{\Gamma(\frac{1}{2}+m-(-1)^r k)}\,q(-m,s), \tag{3.12}$$
$$q(m,s)=\frac{1}{\Gamma(1+2m)}\,e^{-\frac{1}{2}s^2}s^{2m-\frac{3}{2}}\{(\tfrac{1}{2}+m-k)\,{}_1F_1(\tfrac{3}{2}+m-k;1+2m;s^2)$$
$$+(\tfrac{1}{2}A+\tfrac{1}{4}-k)\,{}_1F_1(\tfrac{1}{2}+m-k;1+2m;s^2)\}. \tag{3.13}$$
Here r is an integer.

The second of equations (2.24) allows two equal and opposite A for each p. If the sign of A is reversed, the k defined by (2.29) becomes $1-k$, and these new values of the parameters appear to lead to further independent solutions for $Q(s)$ and $K(s)$. The significance of the transformation
$$A\to -A,\quad k\to 1-k, \tag{3.14}$$
may be most readily appreciated by considering $Q(s)$ in the form (3.10). Kummer's first theorem shows that
$$Q(s\,e^{\frac{1}{2}i\pi},-A,1-k,m)=\left(\frac{\frac{1}{4}+m+\frac{1}{2}A}{\frac{1}{4}+m-\frac{1}{2}A}\right)e^{\pi i(m+\frac{1}{4})}\,Q(s,A,k,m).$$

If the first of (3.1) is now written more explicitly as
$$\Psi(\zeta,A,k,m;\mathscr{C})=\int_{\mathscr{C}}e^{s\zeta}\,Q(s,A,k,m)\,ds,$$
we see that
$$\Psi(\zeta,-A,1-k,m;\mathscr{C})=\left(\frac{\frac{1}{4}+m+\frac{1}{2}A}{\frac{1}{4}+m-\frac{1}{2}A}\right)e^{\pi i(m+\frac{3}{4})}\,\Psi(\zeta\,e^{\frac{1}{2}i\pi},A,k,m;\mathscr{C}_-), \tag{3.15}$$

where $\mathscr{C}_-$ is the contour obtained by rotating $\mathscr{C}$ through a right angle about $s = 0$ in the negative sense. Similar remarks apply to V.

It will be seen later that, for each choice of integrand and contour, there exists a sector of the ζ-plane in which (3.1) provides a convergent solution, and one of the main obstacles encountered in the theory is that of extending a solution from one ζ sector to others. Although it is now clear from the argument just given that (3.14) provides, from a solution valid in one ζ sector, a new solution valid in a different ζ sector (displaced by an angle of $\frac{1}{2}\pi$), the two solutions are not related to each other in any obvious way. For this reason, solutions obtained by the transformation (3.14) will not be examined further.

The problems raised by the choice of the contour in (3.1) will now be considered. The discussion will be general: the treatment of a number of exceptional cases will be deferred to § 5. It is easily shown that the conditions that the integrated parts, obtained in the reduction of (2.25)–(2.27) to (3.2)–(3.4), should vanish are

$$[e^{s\zeta} s^2 Q(s)]_{\mathscr{C}} = [e^{s\zeta} K(s)]_{\mathscr{C}} = 0. \tag{3.16}$$

The only singularity appearing in the integrands of (3.1) is $s = 0$, which is in general $[m \mp \frac{1}{4}(2r-1)$, r an integer$]$ a branch point. Thus in general $\mathscr{C}$ may not encircle $s = 0$, neither may it terminate there [except if $\mathrm{Re}\,(m) > \frac{1}{4}$ and the choice (3.8) is made, with upper sign]. Hence in general $\mathscr{C}$ may terminate only at infinity. To examine this possibility, the following asymptotic expansion for $z \to \infty$ is noted (Slater 1960, eq. 4.1.6):

$$\begin{aligned} {}_1F_1(a;b;z) \simeq \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} e^{-2\pi n i(a-b)}\, {}_2F_0\left(b-a, 1-a;; \frac{1}{z}\right) \\ + \frac{\Gamma(b)}{\Gamma(b-a)} z^{-a} e^{\pi i a(2n+\epsilon)}\, {}_2F_0\left(a, 1+a-b;; -\frac{1}{z}\right). \end{aligned} \tag{3.17}$$

Here n is integral and

$$\epsilon = \left\{\begin{array}{ll} 1, & \text{for} \quad 2n\pi < \arg z < (2n+1)\pi, \\ -1, & \text{for} \quad (2n-1)\pi < \arg z < 2n\pi. \end{array}\right. \tag{3.18}$$

It is clear from (3.17) that, for general k and m, the integrand of (3.1) possesses anti-Stokes lines R_N, given by

$$\arg s = -\tfrac{1}{4}\pi(2N-1), \tag{3.19}$$

and that, between these, the asymptotic expansion of $Q(s)$ contains both $e^{\frac{1}{2}s^2}$ and $e^{-\frac{1}{2}s^2}$ terms, one of which is exponentially large. It is true that, by forming the appropriate linear combination [namely the Q_N of (3.12)], the exponentially large term can be eliminated, but *only* in a *single* sector, namely the domain D_N defined by $\arg s$ lying between the values defining R_N and R_{N+1}. This may be verified explicitly by substituting (3.17) into (3.12), yielding

$$\begin{aligned} Q_{2r}(s) \simeq {} & \frac{2i \sin 2\pi m(n+r)}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)} e^{\frac{1}{2}s^2 - 2\pi n i(\frac{1}{2}-k)} s^{-2k-\frac{1}{2}}\, {}_2F_0\left(-\tfrac{1}{2}+m+k, -\tfrac{1}{2}-m+k;; \frac{1}{s^2}\right) \\ & + \frac{1}{\pi}\{m^2 - (k-\tfrac{1}{2})^2\}\{e^{\pi i m(2n+2r+\epsilon)} \sin\pi(\tfrac{1}{2}-m-k) - e^{-\pi i m(2n+2r+\epsilon)} \sin\pi(\tfrac{1}{2}+m-k)\} \\ & \times e^{-\frac{1}{2}s^2 + (2n+\epsilon)\pi i(\frac{3}{2}-k)} s^{2k-\frac{9}{2}}\, {}_2F_0\left(\tfrac{3}{2}+m-k, \tfrac{3}{2}-m-k;; -\frac{1}{s^2}\right) \\ & + \frac{(\frac{1}{2}A + \frac{1}{4} - k)\, 2i \sin 2\pi m(n+r)}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)} e^{\frac{1}{2}s^2 + 2\pi n i(\frac{1}{2}+k)} s^{-2k-\frac{5}{2}}\, {}_2F_0\left(\tfrac{1}{2}+m+k, \tfrac{1}{2}-m+k;; \frac{1}{s^2}\right) \\ & + \frac{1}{\pi}(\tfrac{1}{2}A + \tfrac{1}{4} - k)\{e^{\pi i m(2n+2r+\epsilon)} \sin\pi(\tfrac{1}{2}-m-k) - e^{-\pi i m(2n+2r+\epsilon)} \sin\pi(\tfrac{1}{2}+m-k)\} \\ & \times e^{-\frac{1}{2}s^2 + (2n+\epsilon)\pi i(\frac{1}{2}-k)} s^{2k-\frac{5}{2}}\, {}_2F_0\left(\tfrac{1}{2}+m-k, \tfrac{1}{2}-m-k;; -\frac{1}{s^2}\right), \end{aligned} \tag{3.20}$$

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and

$$Q_{2r-1}(s) \simeq \frac{1}{\pi}\{e^{\pi i m(2n+2r-1)}\sin\pi(\tfrac{1}{2}-m+k)-e^{-\pi i m(2n+2r-1)}\sin\pi(\tfrac{1}{2}+m+k)\}\,e^{\frac{1}{2}s^2-2\pi n i(\frac{1}{2}-k)}\,s^{-2k-\frac{1}{2}}$$
$$\times {}_2F_0\left(-\tfrac{1}{2}+m+k,\,-\tfrac{1}{2}-m+k;;\frac{1}{s^2}\right)+\frac{\{m^2-(k-\frac{1}{2})^2\}\,2i\sin\pi m(2n+2r+\epsilon-1)}{\Gamma(\frac{1}{2}-m+k)\,\Gamma(\frac{1}{2}+m+k)}$$
$$\times e^{-\frac{1}{2}s^2+(2n+\epsilon)\pi i(\frac{3}{2}-k)}\,s^{2k-\frac{9}{2}}\,{}_2F_0\left(\tfrac{3}{2}+m-k,\tfrac{3}{2}-m-k;;-\frac{1}{s^2}\right)+\frac{1}{\pi}\left\{\frac{A}{2}+\tfrac{1}{4}-k\right\}$$
$$\times\{e^{\pi i m(2n+2r-1)}\sin\pi(\tfrac{1}{2}-m+k)-e^{-\pi i m(2n+2r-1)}\sin\pi(\tfrac{1}{2}+m+k)\}\,e^{\frac{1}{2}s^2+2\pi n i(\frac{1}{2}+k)}\,s^{-2k-\frac{5}{2}}$$
$$\times {}_2F_0\left(\tfrac{1}{2}+m+k,\tfrac{1}{2}-m+k;;\frac{1}{s^2}\right)+\frac{(\frac{1}{2}A+\frac{1}{4}-k)\,2i\sin\pi m(2n+2r+\epsilon-1)}{\Gamma(\frac{1}{2}-m+k)\,\Gamma(\frac{1}{2}+m+k)}$$
$$\times e^{-\frac{1}{2}s^2+(2n+\epsilon)\pi i(\frac{1}{2}-k)}\,s^{2k-\frac{5}{2}}\,{}_2F_0\left(\tfrac{1}{2}+m-k,\tfrac{1}{2}-m-k;;-\frac{1}{s^2}\right), \tag{3.21}$$

where now, by (3.18),
$$\epsilon=\begin{cases}1, & \text{for}\quad n\pi<\arg s<(n+\frac{1}{2})\pi,\\ -1, & \text{for}\quad (n-\frac{1}{2})\pi<\arg s<n\pi.\end{cases} \tag{3.22}$$

(The corresponding expansions for $s^{-1}K(s)$ may be derived by reversing the signs of the terms involving the factor $\frac{1}{2}A+\frac{1}{4}-k$; cf. (3.10) and (3.11).) Even by selecting Q_N, we would only obtain the trivial zero solution if we allowed *both* ends of $\mathscr{C}$ to terminate in D_N. In fact, no matter what choice of Q is made, at least one end of $\mathscr{C}$ must approach infinity along one of the rays (3.19). Then by (3.17), the $Q(s)$ and $K(s)$ of (3.1) are at most algebraically large, and the $e^{\zeta s}$ term decides which of the rays (3.19) is permissible. It is found that $\mathscr{C}$ may approach infinity along R_N if, for some integer n,
$$(2n+\tfrac{1}{2}N+\tfrac{1}{4})\,\pi<\arg\zeta<(2n+\tfrac{1}{2}N+\tfrac{5}{4})\,\pi. \tag{3.23}$$

Hence every solution (3.1) is valid in only a limited sector of the ζ-plane. This fact gives rise to unusual difficulty in relating solutions in different ζ sectors.

Solutions may be found, for any Q, by allowing $\mathscr{C}$ to approach infinity along two different rays R_N. For instance, solutions valid in the range (3.23) may be obtained by selecting the contour C_N, defined as starting from infinity on R_N and ending at infinity on R_{N-4}, having passed round $s=0$ in the positive sense (C_1 is shown in figure 1). In order to find the complete solution of (2.27), four independent solutions must be obtained for a given $\arg\zeta$. It might be thought that the C_N contours would suffice since, for any particular $\arg\zeta$, two such contours are acceptable, and for each of these, two independent choices of Q are available. Unfortunately, however, as we will show later, the four solutions obtained in this way are always linearly dependent. It is necessary to make use of solutions employing Q_N and having one end terminating in D_N, as described above. Many choices of four independent solutions are possible. Considerations of symmetry make it convenient to proceed as follows.

Let the contour γ_N^+ start at infinity on R_N, pass to the left of the origin, and end at infinity somewhere in D_{N+1}; similarly, let γ_N^- pass to the right of the origin and terminate in D_{N-2}. Let the contour λ_N^+ start at infinity on R_N, pass to the left of the origin, but end at infinity somewhere in D_{N+2}; similarly, let λ_N^- pass to the right of the origin and terminate in D_{N-3}. All these contours are shown in figure 1 for the case $N=1$. When ζ lies in the sector (3.23), solutions may be obtained from all the γ_N and λ_N (suffices, mod 4) contours, the corresponding integrands being the appropriate $Q_r(s)$ of (3.12).

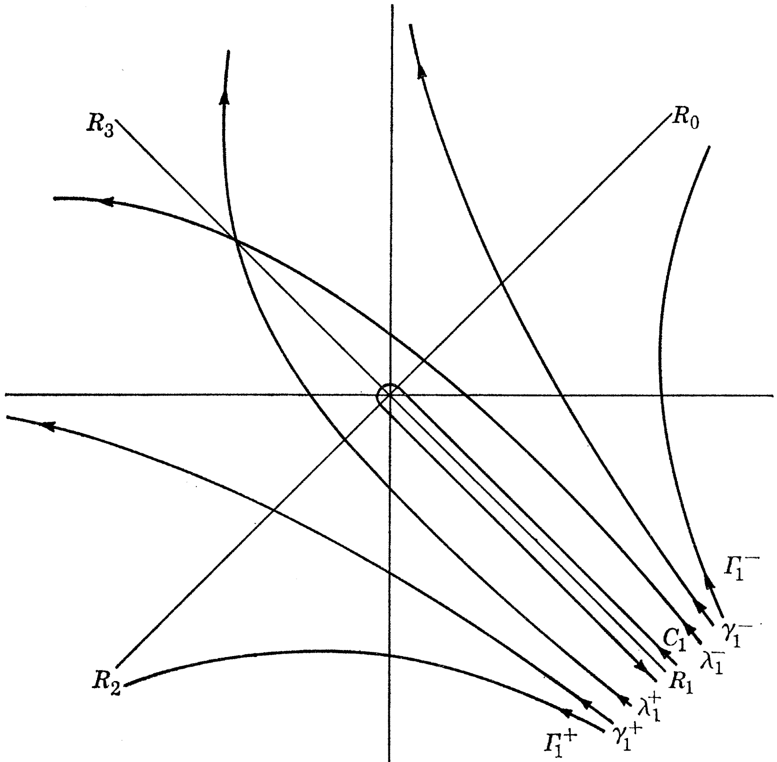

FIGURE 1. Contours from R_1 in the s-plane.

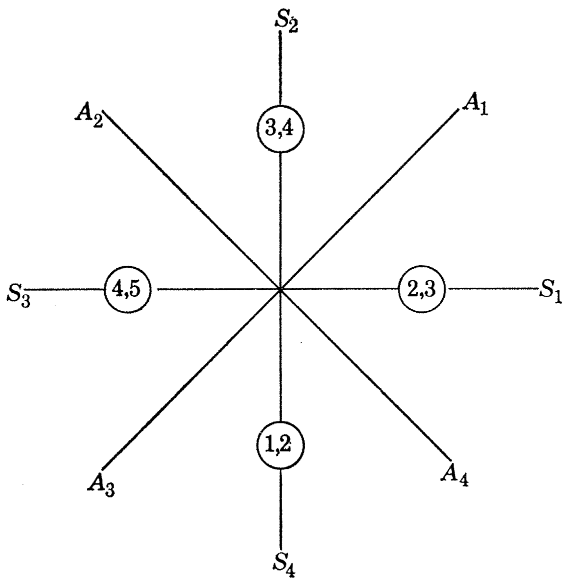

FIGURE 2. Stokes and anti-Stokes lines in the ζ-plane.

Let the contour Γ_N^+ start at infinity on R_N, remain in sector D_N, and end at infinity on R_{N+1}; similarly, let Γ_N^- pass through D_{N-1} from R_N to R_{N-1}. (Thus Γ_N^+ and Γ_{N+1}^- are essentially the same, though described in the opposite sense; Γ_1^+ and Γ_1^- are depicted in figure 1.) When ζ lies in the sector

$$(2n+\tfrac{1}{2}N+\tfrac{3}{4})\,\pi < \arg\zeta < (2n+\tfrac{1}{2}N+\tfrac{5}{4})\,\pi, \tag{3.24}$$

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where n is an integer, solutions may be obtained using Γ_N^+. Moreover, for this sector all the C_r, γ_r and λ_r solutions are available both for $r = N$ and $r = N+1$: this is illustrated in figure 2 by the circled numbers $(N, N+1)$ in each sector of the ζ plane; as before, the suffices are arbitrary to a multiple of 4.

The next task is to relate solutions in adjacent ζ sectors. To this end, the following notation will be used:

$$\Psi_r(\zeta; \mathscr{C}_N) = \int_{\mathscr{C}_N} e^{s\zeta} Q_r(s)\, ds. \tag{3.25}$$

Here $\mathscr{C}_N$ is any of the contours defined above in association with R_N, and it is assumed implicitly that $\arg \zeta$ is such that the integral converges, the appropriate integrand also (in the case of the γ_N and λ_N contours) being selected. It is worth noting that $\Psi_r(\zeta)$ is an integral function of ζ, the substitution $s = s'\, e^{2\pi i l}$ in (3.25) leading, for integral l, to the result

$$\Psi_r(\zeta; \mathscr{C}_N) = e^{-\pi i l}\, \Psi_{r+4l}(\zeta; \mathscr{C}_{N+4l}). \tag{3.26}$$

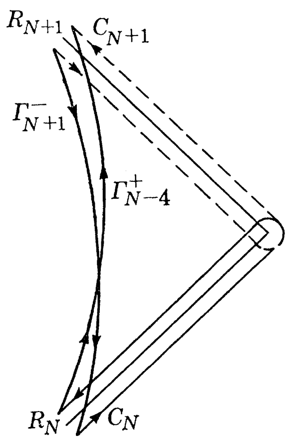


FIGURE 3. Distortion of C_{N+1} contour in the s-plane.

The solutions arising from Γ_N^+ have an interesting property which is, in fact, central to the later discussion. Since there are only two possible independent choices of integrand, the contour Γ_N^+ can at most provide two independent solutions in (3.24). Now (3.12) may be selected as one of these integrands, and, since this is exponentially small at infinity in D_N, the contour Γ_N^+, when deformed to infinity, clearly yields the trivial zero solution. In other words, the solutions defined by Γ_N^+ are, for all choices of $Q(s)$, proportional. This important property may be used, for example, to show that the C_N contours cannot provide four independent $\Psi(\zeta)$. For $\arg \zeta$ in (3.24), these solutions are linear combinations of $\Psi_r(\zeta; C_s)$ and $\Psi_{r-1}(\zeta; C_s)$ where $s = N$ or $N+1$, and r is given any integral value. From figure 3 it is clear that

$$C_{N+1} \equiv \Gamma_{N+1}^- + C_N + \Gamma_{N-4}^+,$$

so that
$$\Psi_r(\zeta; C_{N+1}) = -\Psi_r(\zeta; \Gamma_N^+) + \Psi_r(\zeta; C_N) + \psi_r(\zeta; \Gamma_{N-4}^+),$$

and similarly when r is replaced by $r-1$. But the solutions $\Psi_r(\zeta; \Gamma_{N(\mathrm{mod}\,4)}^+)$ and $\Psi_{r-1}(\zeta; \Gamma_{N(\mathrm{mod}\,4)}^+)$ are proportional. It follows that all four C_N solutions can be expressed as linear combinations of the three solutions $\Psi_r(\zeta; C_N)$, $\Psi_{r-1}(\zeta; C_N)$, and $\Psi_r(\zeta; \Gamma_N^+)$.

The four γ_N and λ_N solutions associated with the ray R_N provide four linearly independent solutions in sector (3.23); for the smaller sector (3.24), two such sets of solutions are associated with the rays R_N and R_{N+1}. (The corresponding contours are illustrated in figure 4.) In order to be able to extend any particular solution to any arg ζ, it is necessary to be able to relate the solutions associated with R_N to those associated with R_{N+1}, i.e. it is necessary to evaluate a 4×4 matrix, T, such that

$$\boldsymbol{s}^N = T\boldsymbol{s}^{N+1}. \tag{3.27}$$

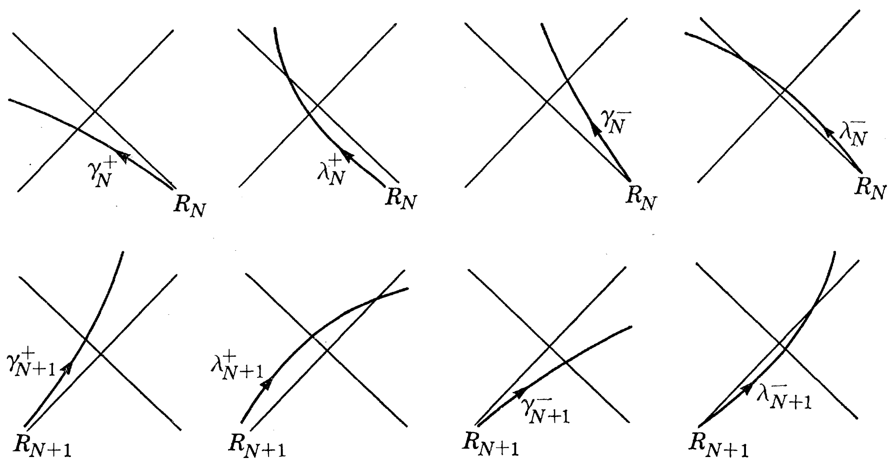


FIGURE 4. Contours in the s-plane useful when $(2n+\frac{1}{2}N+\frac{3}{4})\,\pi < \arg\zeta < (2n+\frac{1}{2}N+\frac{5}{4})\,\pi$.

Here $\boldsymbol{s}^N$ is the column vector of solutions defined by

$$\boldsymbol{s}^N = \begin{bmatrix} s_1^N \\ s_2^N \\ s_3^N \\ s_4^N \end{bmatrix} = \begin{bmatrix} \Psi_{N+1}(\zeta;\gamma_N^+) \\ \Psi_{N+2}(\zeta;\lambda_N^+) \\ \Psi_{N-2}(\zeta;\gamma_N^-) \\ \Psi_{N-3}(\zeta;\lambda_N^-) \end{bmatrix}. \tag{3.28}$$

The matrix T may be found from the properties of the solutions already described, and from (3.12), (3.13), (3.20) and (3.21). Since the asymptotic form of $Q_r(s)$ depends on whether r is even or odd, the two cases of N even and N odd must be treated separately: consider first the former.

The contour γ_N^+ used in the definition of s_1^N may be deformed into Γ_N^+, since the integrand dies exponentially both in D_{N+1} and on R_{N+1}; cf. (3.25). In the same way, the contour γ_{N+1}^- defining s_3^{N+1} may be deformed into Γ_N^+ taken in the reverse direction. Thus, by the fundamental property of the Γ_N solutions, s_1^N and s_3^{N+1} are proportional. The constant of proportionality may be found from (3.21) in the special cases of $r = \frac{1}{2}N$ and $r = \frac{1}{2}N+1$; in both we must take $n = -\frac{1}{2}N$, since $-(\frac{1}{2}N+\frac{1}{4})\,\pi \leqslant \arg s \leqslant -(\frac{1}{2}N-\frac{1}{4})\,\pi$ on Γ_N^+. The terms in $e^{-\frac{1}{2}s^2}$ may be ignored since (for these) the contour may be deformed to infinity, and zero contributions to the integrals result. The required constant of proportionality is therefore minus the ratio of the $e^{\frac{1}{2}s^2}$ coefficients, i.e. $-e^{-2\pi ik}$. Thus the first row of T is $[0, 0, e^{-2\pi ik}, 0]$.

The contour λ_N^+ used in the definition of s_2^N may be deformed into $\Gamma_N^+ + \gamma_{N+1}^+$, and therefore

$$s_2^N = s_1^{N+1} + Ks_3^{N+1},$$

the constant K being evaluated as in the previous case. Defining $f(k)$ by

$$f \equiv f(k) = \frac{2\pi}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)}, \tag{3.29}$$

a short calculation shows that $K = f e^{-\pi ik}$. Thus the second row of T is $[1, 0, f e^{-\pi ik}, 0]$.

The contour γ_N^- used in the definition of s_3^N may be deformed into $\Gamma_N^+ + \lambda_{N+1}^-$, and a similar calculation gives the third row of T as $[0, 0, -f\,\mathrm{e}^{-\pi ik}, 1]$.

The contour λ_N^- used in the definition of s_4^N is more difficult to treat, and the contour C_{N+1} defined after (3.23) will be used. Since λ_N^- may be deformed into $\Gamma_N^+ + C_{N+1}$, we have

$$s_4^N = K' s_3^{N+1} + \Psi_{N-3}(\zeta; C_{N+1}), \tag{3.30}$$

where the constant K' may be evaluated as K was. Since

$$C_{N+1} \equiv \gamma_{N+1}^- - \lambda_{N-3}^+ \equiv \lambda_{N+1}^- - \gamma_{N-3}^+,$$

(3.26) gives $$\Psi_{N-1}(\zeta; C_{N+1}) = s_2^{N+1} + s_3^{N+1}, \quad \Psi_{N-2}(\zeta; C_{N+1}) = s_1^{N+1} + s_4^{N+1}. \tag{3.31}$$

Moreover, the definition (3.12) of $Q_r(s)$ gives

$$\Psi_{N-3}(\zeta; C_{N+1}) = -\mathrm{e}^{2\pi ik}\,\Psi_{N-1}(\zeta; C_{N+1}) + f(-k)\,\mathrm{e}^{\pi ik}\,\Psi_{N-2}(\zeta; C_{N+1}). \tag{3.32}$$

On combining (3.30) to (3.32), we obtain

$$s_4^N = f(-k)\,\mathrm{e}^{\pi ik}\,s_1^{N+1} - \mathrm{e}^{2\pi ik}\,s_2^{N+1} - 2(\cos 2\pi m + \cos 2\pi k)\,s_3^{N+1} + f(-k)\,\mathrm{e}^{\pi ik}\,s_4^{N+1},$$

which yields the final row of T.

On repeating these calculations in the case of odd N, the same matrix T is obtained, except that k is everywhere replaced by $-k$. We define, therefore, $T(k)$ by

$$T(k) = \begin{bmatrix} 0 & 0 & \mathrm{e}^{-2\pi ik} & 0 \\ 1 & 0 & f(k)\,\mathrm{e}^{-\pi ik} & 0 \\ 0 & 0 & -f(k)\,\mathrm{e}^{-\pi ik} & 1 \\ f(-k)\,\mathrm{e}^{\pi ik} & -\mathrm{e}^{2\pi ik} & -f(k)f(-k) & f(-k)\,\mathrm{e}^{\pi ik} \end{bmatrix}, \tag{3.33}$$

and write (3.27) in the form $$\mathbf{s}^N = T(k(-1)^N)\,\mathbf{s}^{N+1}. \tag{3.34}$$

It is sometimes useful to express this in the inverse form

$$\mathbf{s}^{N+1} = [T(k(-1)^N)]^{-1}\,\mathbf{s}^N, \tag{3.35}$$

where, as can readily be verified,

$$[T(k)]^{-1} = \begin{bmatrix} -f(k)\,\mathrm{e}^{\pi ik} & 1 & 0 & 0 \\ -f(k)f(-k) & f(-k)\,\mathrm{e}^{-\pi ik} & f(-k)\,\mathrm{e}^{-\pi ik} & -\mathrm{e}^{-2\pi ik} \\ \mathrm{e}^{2\pi ik} & 0 & 0 & 0 \\ f(k)\,\mathrm{e}^{\pi ik} & 0 & 1 & 0 \end{bmatrix}. \tag{3.36}$$

It is interesting to note that $[T(k)]^{-1}$ is $[T(k)]^*$, relabelled according to $(1, 2, 3, 4) \rightarrow (3, 4, 1, 2)$. It is also worth observing that, on applying (3.34) four times, we obtain

$$\mathbf{s}^N = [T(k(-1)^N)\,T(k(-1)^{N+1})]^2\,\mathbf{s}^{N+4} = -\mathbf{s}^{N+4}, \tag{3.37}$$

which agrees with the more general result (3.26), and at the same time provides a useful check on $T(k)$.

4. Asymptotic expansions for large $|\zeta|$; the saddle points

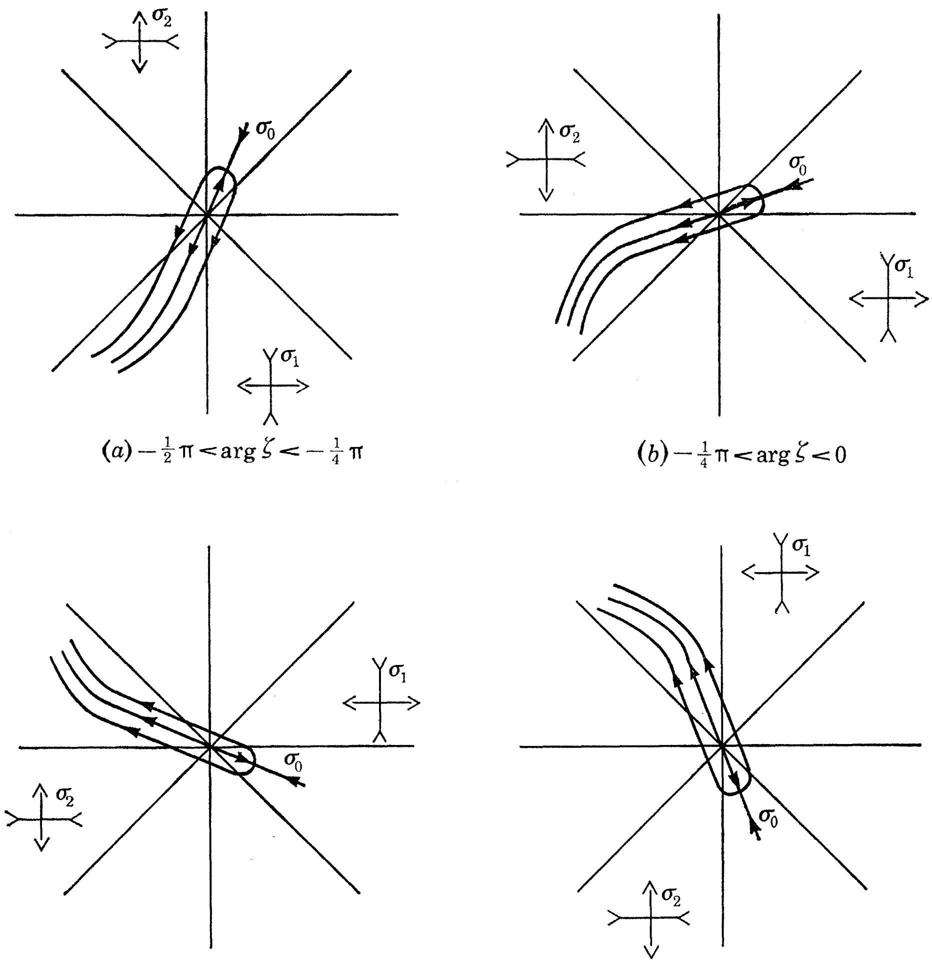
It is readily seen from (3.20) and (3.21) that the integrands of solutions (3.1) have saddle points, located asymptotically at

$$s = \sigma_1 = \zeta \,\mathrm{e}^{2\pi i l} \quad \text{and} \quad s = \sigma_2 = \zeta \,\mathrm{e}^{\pi i(2l+1)}, \tag{4.1}$$

where l is an integer. These two saddle points will be referred to as 'outer saddle points' since an inspection of (3.12) for small $|s|$ reveals the presence of another 'inner saddle point' at

$$s = \sigma_0 = \frac{\frac{3}{2}+2m}{\zeta}\,\mathrm{e}^{2\pi i l}. \tag{4.2}$$

It will be later realized that the outer saddle points are associated with the resistive solutions (2.28), while the inner saddle point is related to the ideal solutions. The inclusion of the integer l in (4.1) and (4.2) is necessary in subsequent calculations since arg s is carefully specified at infinity on each contour used, and is therefore uniquely defined at any saddle point over which the (suitably distorted) contour may pass.



(a) $-\frac{1}{2}\pi < \arg \zeta < -\frac{1}{4}\pi$ (b) $-\frac{1}{4}\pi < \arg \zeta < 0$

(c) $0 < \arg \zeta < \frac{1}{4}\pi$ (d) $\frac{1}{4}\pi < \arg \zeta < \frac{1}{2}\pi$

Figure 5 (*a* to *d*). For legend see facing page.

The contribution to the expression (3.1) for $\Psi(\zeta)$ from the neighbourhood of each saddle point, when $\mathscr{C}$ is deformed to pass over it along the steepest curve, will now be evaluated to leading order. This means that the second and third terms in (3.20) and (3.21) are neglected, and that in the remaining terms the leading term only (i.e. unity) is included from each ${}_2F_0$ function.

(a) *The saddle point* σ_1

The contour $\mathscr{C}$ is deformed to pass over $s = \sigma_1$ in the steepest direction from $\arg(s-\sigma_1) = \pi$ to $\arg(s-\sigma_1) = 0$. When $Q_{2r}(s)$ is used in the integrand, the contribution to the integral is

$$\sqrt{\frac{2}{\pi}}\left(\frac{A}{2}+\frac{1}{4}-k\right)\{\mathrm{e}^{\pi m\mathrm{i}(2n+2r+\epsilon)}\sin\pi(\tfrac{1}{2}-m-k)-\mathrm{e}^{-\pi m\mathrm{i}(2n+2r+\epsilon)}\sin\pi(\tfrac{1}{2}+m-k)\}$$
$$\times(\zeta\,\mathrm{e}^{2\pi\mathrm{i}l})^{2k-\frac{5}{2}}\,\mathrm{e}^{(2n+\epsilon)\pi\mathrm{i}(\frac{1}{2}-k)+\frac{1}{2}\zeta^2}, \qquad (4.3)$$

where n and ϵ are given by (3.22) for the value of $\arg\sigma_1$ indicated. Similarly, for $Q_{2r-1}(s)$, the contribution is

$$\sqrt{(2\pi)}\,\frac{(\frac{1}{2}A+\frac{1}{4}-k)\,2\mathrm{i}\sin\pi m(2n+2r+\epsilon-1)}{\Gamma(\frac{1}{2}-m+k)\,\Gamma(\frac{1}{2}+m+k)}\,(\zeta\,\mathrm{e}^{2\pi\mathrm{i}l})^{2k-\frac{5}{2}}\,\mathrm{e}^{(2n+\epsilon)\pi\mathrm{i}(\frac{1}{2}-k)+\frac{1}{2}\zeta^2}. \qquad (4.4)$$

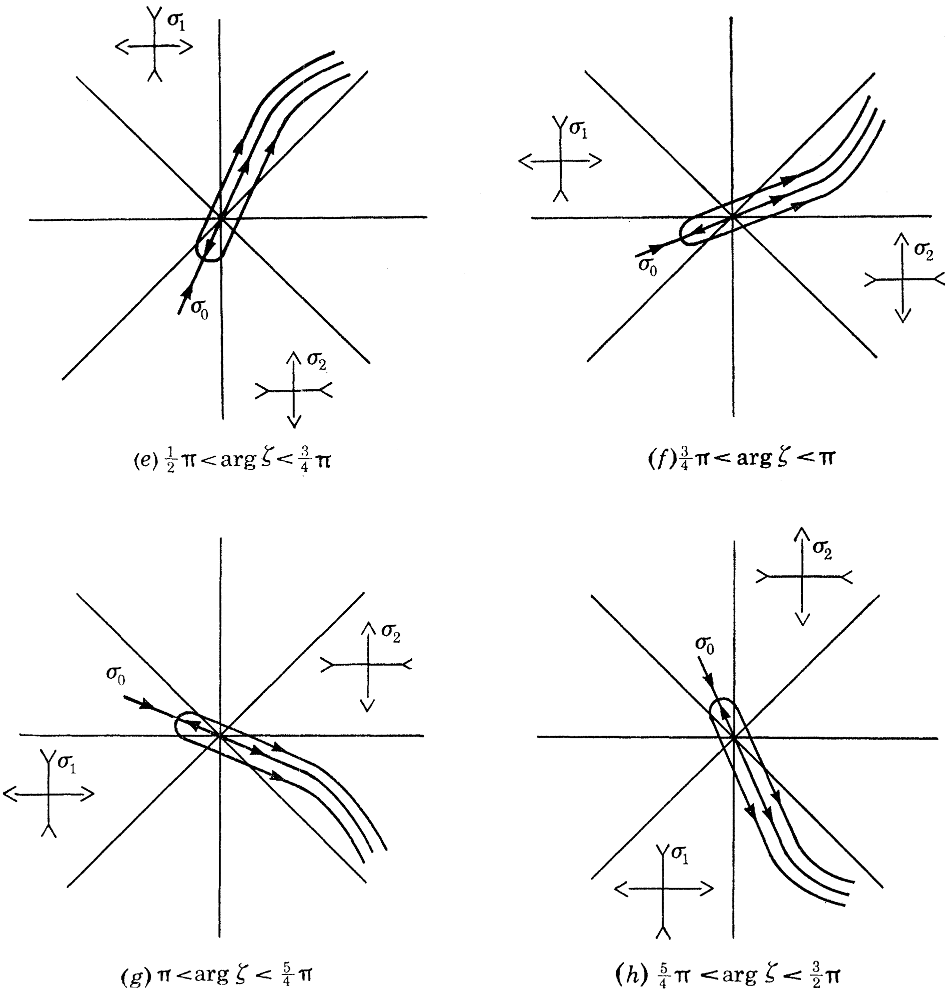


FIGURE 5 (*e* to *h*). Steepest curves in the s-plane over the inner saddle point, σ_0, for a number of ranges of $\arg\zeta$.

(b) *The saddle point* σ_2

The contour is deformed to pass over $s = \sigma_2$ in the steepest direction from $\arg(s-\sigma_2) = \frac{1}{2}\pi$ to $\arg(s-\sigma_2) = -\frac{1}{2}\pi$. The contributions made to the integral, when $Q_{2r}(s)$ and $Q_{2r-1}(s)$ are used, are respectively

$$\sqrt{(2\pi)}\,\frac{2\sin 2\pi m(n+r)}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)}\,(\zeta\,\mathrm{e}^{\pi\mathrm{i}(2l+1)})^{-2k-\frac{1}{2}}\,\mathrm{e}^{-2\pi n\mathrm{i}(\frac{1}{2}-k)-\frac{1}{2}\zeta^2}, \tag{4.5}$$

$$-\mathrm{i}\sqrt{\frac{2}{\pi}}\{\mathrm{e}^{\pi m\mathrm{i}(2n+2r-1)}\sin\pi(\tfrac{1}{2}-m+k)-\mathrm{e}^{-\pi m\mathrm{i}(2n+2r-1)}\sin\pi(\tfrac{1}{2}+m+k)\}$$
$$\times(\zeta\,\mathrm{e}^{\pi\mathrm{i}(2l+1)})^{-2k-\frac{1}{2}}\,\mathrm{e}^{-2\pi n\mathrm{i}(\frac{1}{2}-k)-\frac{1}{2}\zeta^2}. \tag{4.6}$$

(c) *The saddle point* σ_0

For real m, the curves of steepest descent from σ_0 are perpendicular to the ray $\arg s = \arg\sigma_0$, but both are asymptotically anti-parallel to it when $|\sigma_0| \ll |s| \ll 1$; see figures 5. If any segment of the contour $\mathscr{C}$ of (3.1) has to be brought into the $|s| \ll 1$ neighbourhood of the origin and over σ_0, the corresponding contributions to the integrals may be derived, to leading order, by neglecting the factor $\mathrm{e}^{-\frac{1}{2}s^2}$ in (3.13) and retaining the leading (i.e. unit) terms of the ${}_1F_1$ functions, and by extending the segment to infinity along the rays $\arg s = (2l\pm 1)\,\pi-\arg\zeta$. The error introduced by this change of contour is exponentially small. The modified contour, $\mathscr{C}_s$, is defined to pass from infinity on the ray $\arg s = (2l-1)\,\pi-\arg\zeta$, encircle the origin in the positive sense, and return to infinity along the ray $\arg s = (2l+1)\,\pi-\arg\zeta$. It is now necessary to evaluate the integral

$$\int_{\mathscr{C}_s}\mathrm{e}^{s\zeta}\,s^{-\frac{3}{2}-2m}\,\mathrm{d}s.$$

This may be cast into a more familiar form by making the substitution $s = -(t/\zeta)\,\mathrm{e}^{2\pi\mathrm{i}l}$ and adopting the convention that $\ln(-t)$ is real when t is on the negative real axis. The contour $\mathscr{C}_s$ transforms into a contour, $\mathscr{C}_t$, which starts and finishes at infinity on the positive t-axis and encircles the origin once counterclockwise. The integral now becomes

$$-\left(\frac{\mathrm{e}^{2\pi\mathrm{i}l}}{\zeta}\right)^{-\frac{1}{2}-2m}\int_{\mathscr{C}_t}\mathrm{e}^{-t}\,(-t)^{-\frac{3}{2}-2m}\,\mathrm{d}t.$$

On referring to Hankel's representation of the gamma function (see, for example, Whittaker & Watson 1927, § 12.22), we can re-express this as

$$\frac{2\pi\mathrm{i}}{\Gamma(\frac{3}{2}+2m)}\left(\frac{\mathrm{e}^{2\pi\mathrm{i}l}}{\zeta}\right)^{-\frac{1}{2}-2m}.$$

In evaluating the contribution made by $Q_r(s)$ to $\Psi(\zeta)$, we find that integrals arise which contain the term $s^{-\frac{3}{2}+2m}$, rather than the term $s^{-\frac{3}{2}-2m}$ appearing above. These possess a different inner saddle point, given by (4.2) with the opposite sign for m. If $m > \frac{3}{4}$, this lies on the opposite side of the origin to σ_0, and the topology of the steepest curves is different. Nevertheless, the contribution it makes to the integrals (3.1) can in all cases be obtained immediately by reversing the sign of m in the expressions given above. In this way, the final contribution made by $Q_r(s)$ is found to be

$$E_r \equiv L_r(m,\zeta)-L_r(-m,\zeta), \tag{4.7}$$

where
$$L_r(m,\zeta) = \frac{2\pi\mathrm{i}(\frac{1}{2}A+\frac{3}{4}+m-2k)}{\Gamma(1+2m)\,\Gamma(\frac{3}{2}-2m)\,\Gamma(\frac{1}{2}-m-(-1)^r k)}\,\mathrm{e}^{\pi\mathrm{i}rm}\left(\frac{\mathrm{e}^{2\pi\mathrm{i}l}}{\zeta}\right)^{-\frac{1}{2}+2m}. \tag{4.8}$$

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For imaginary m, the saddle point σ_0 is displaced by up to $\frac{1}{2}\pi$ radians from the ray

$$\arg s = -\arg \zeta,$$

and the level curves are correspondingly distorted near the origin. The curves of steepest descent, however, remain asymptotically anti-parallel to the ray $\arg s = -\arg\zeta$ when $|\sigma_0| \ll |s| \ll 1$, so that the results obtained above remain valid.

Now that the contours for the basic set of solutions have been defined and the contributions from the saddle points have been obtained, expansions for large $|\zeta|$ may be listed. The range

$$-\tfrac{1}{2}\pi < \arg\zeta < \tfrac{3}{2}\pi \tag{4.9}$$

has been selected for detailed consideration, although any other 2π range for $\arg\zeta$ would have been equally acceptable. Figures 5 show, for a number of values of $\arg\zeta$ in the range (4.9), the inner saddle point σ_0 and the steepest curves through it, and also the outer saddle points with the directions of their steepest curves. It may be seen that, in addition to the natural division (3.23) of (4.9), there is a change in the topology of the inner curves as ζ crosses any of the lines S_N depicted in figure 2. In fact, these are the Stokes lines, and the asymptotic expansions must be considered separately in each of the $\frac{1}{4}\pi$ intervals of $\arg\zeta$ indicated in figures 5. The process is considered in detail for $\boldsymbol{s}^2$ in the first subinterval of figure 5(*a*). Results for other intervals are listed in table 1.

In the sector $-\frac{1}{2}\pi < \arg\zeta < -\frac{1}{4}\pi$, the contour γ_2^+ may be deformed away from the origin to pass over the saddle point σ_2 and to leading order the solution s_1^2 is therefore given by (4.6) with $l = n = \epsilon = -1$ and $r = 2$. The contour λ_2^+ may be deformed to pass first round the contour C_6 in the reverse direction and then from infinity on R_6 to infinity in the required sector D_4, passing over the saddle point σ_1 in the process. The leading contributions to s_2^2 are therefore given by (4.7) with $l = -1$ and $r = 4$, and (4.3) with $l = \epsilon = -1$ and $r = -n = 2$. The contour γ_2^- may be deformed away from the origin over the saddle point σ_1, the resulting leading order contribution to s_3^2 being given by (4.3) with $l = n = r = 0$ and $\epsilon = -1$. Finally, the contour λ_2^- may be first of all deformed into C_2 and then from infinity on R_{-2} to infinity in the required sector D_{-1} over the saddle point σ_2. The leading contributions to s_4^2 are given by (4.7) with $l = 0$ and $r = -1$, and (4.6) with $l = r = 0$ and $n = -\epsilon = 1$.

The terms in the solutions of table 1 have been arranged in order of decreasing dominancy, and the solutions show that the lines S_N of figure 2 are indeed the Stokes lines, i.e. the subdominant parts of the solutions may change on passing across them. The lines A_N are the anti-Stokes lines across which the order of dominancy reverses.

The results of table 1, together with the relations (3.34) and (3.35), may now be used to list four independent solutions in the complete range (4.9) of $\arg\zeta$. The following solutions, denoted by $s(i)$, have been chosen as convenient to extend to the full range:

$$\left.\begin{aligned} s(1) &= s_1^3 &&\text{in}\quad 0 < \arg\zeta < \tfrac{1}{4}\pi,\\ s(2) &= s_3^3 &&\text{in}\quad 0 < \arg\zeta < \tfrac{1}{4}\pi,\\ s(3) &= s_1^4 &&\text{in}\quad \tfrac{3}{4}\pi < \arg\zeta < \pi,\\ s(4) &= s_1^3 + s_4^3 &&\text{in}\quad 0 < \arg\zeta < \tfrac{1}{4}\pi.\end{aligned}\right\} \tag{4.10}$$

The full asymptotic expansions of these solutions have been displayed in table 2. In each case, the terms are in the order of decreasing dominancy for the first half of the interval named (i.e. for the smaller values of $\arg\zeta$), and in order of increasing dominancy for the remainder (larger $\arg\zeta$).

Table 1. Solutions associated with the rays R_N for $-\frac{1}{2}\pi < \arg\zeta < \frac{3}{2}\pi$

Notation

$f = f(k)$; see (3.29), $g = \frac{1}{2}A + \frac{1}{4} - k$, $h = f(-k)$,

$\tilde{F} = \sqrt{\frac{2}{\pi}} \sin 2\pi m\, \zeta^{-2k-\frac{1}{2}}$, $\tilde{G} = \sqrt{\frac{2}{\pi}} \sin 2\pi m\, \zeta^{2k-\frac{5}{2}}$, $H = \frac{1}{2}\zeta^2$,

E_r is the $l = 0$ case of (4.7).

Sector 1. $-\frac{1}{2}\pi < \arg\zeta < -\frac{1}{4}\pi$

R_2 solutions: $s_1^2 \simeq \mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$, $s_2^2 \simeq E_0 + g\tilde{G}\,\mathrm{e}^{H}$,
$s_3^2 \simeq -g\tilde{G}\,\mathrm{e}^{H}$, $s_4^2 \simeq -\mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + E_{-1}$.

Sector 2. $-\frac{1}{4}\pi < \arg\zeta < 0$

R_2 solutions as in sector 1, with dominancy reversed.

R_3 solutions:
$s_1^3 \simeq g\tilde{G}\,\mathrm{e}^{H} + E_0 - \mathrm{i}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_2^3 \simeq E_1 - \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}(1 + 2\,\mathrm{e}^{-2\pi\mathrm{i}k}\cos 2\pi m)\,\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^3 \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_4^3 \simeq -g\tilde{G}\,\mathrm{e}^{H} + \mathrm{i}f\tilde{F}\,\mathrm{e}^{-H}$.

Sector 3. $0 < \arg\zeta < \frac{1}{4}\pi$

R_2 solutions:
$s_1^2 \simeq \mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_2^2 \simeq g\tilde{G}\,\mathrm{e}^{H} + \mathrm{i}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^2 \simeq -g\tilde{G}\,\mathrm{e}^{H} + E_0 - \mathrm{i}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_4^2 \simeq E_{-1} - \mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}k}(1 + 2\,\mathrm{e}^{2\pi\mathrm{i}k}\cos 2\pi m)\tilde{F}\,\mathrm{e}^{-H}$.

R_3 solutions: $s_1^3 \simeq g\tilde{G}\,\mathrm{e}^{H}$, $s_2^3 \simeq E_1 - \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^3 \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$, $s_4^3 \simeq -g\tilde{G}\,\mathrm{e}^{H} + E_0$.

Sector 4. $\frac{1}{4}\pi < \arg\zeta < \frac{1}{2}\pi$

R_3 solutions as in sector 3, with dominancy reversed.

R_4 solutions:
$s_1^4 \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + E_1 - \mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$,
$s_2^4 \simeq E_2 - \mathrm{e}^{-2\pi\mathrm{i}k}(1 + 2\,\mathrm{e}^{2\pi\mathrm{i}k}\cos 2\pi m)g\tilde{G}\,\mathrm{e}^{H}$,
$s_3^4 \simeq \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}$,
$s_4^4 \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + \mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$.

Sector 5. $\frac{1}{2}\pi < \arg\zeta < \frac{3}{4}\pi$

R_3 solutions:
$s_1^3 \simeq g\tilde{G}\,\mathrm{e}^{H}$,
$s_2^3 \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + \mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$,
$s_3^3 \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + E_1 - \mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$,
$s_4^3 \simeq E_0 - (1 + 2\,\mathrm{e}^{-2\pi\mathrm{i}k}\cos 2\pi m)g\tilde{G}\,\mathrm{e}^{H}$.

R_4 solutions: $s_1^4 \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$, $s_2^4 \simeq E_2 - \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}$,
$s_3^4 \simeq \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}$, $s_4^4 \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H} + E_1$.

Sector 6. $\frac{3}{4}\pi < \arg\zeta < \pi$

R_4 solutions as in sector 5, with dominancy reversed.

R_5 solutions:
$s_1^5 \simeq -\mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H} + E_2 + \mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}k}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_2^5 \simeq E_3 + \mathrm{i}\,\mathrm{e}^{3\pi\mathrm{i}k}(1 + 2\,\mathrm{e}^{-2\pi\mathrm{i}k}\cos 2\pi m)\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^5 \simeq -\mathrm{i}\,\mathrm{e}^{3\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_4^5 \simeq \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H} - \mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}k}f\tilde{F}\,\mathrm{e}^{-H}$.

Sector 7. $\pi < \arg\zeta < \frac{5}{4}\pi$

R_4 solutions:
$s_1^4 \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_2^4 \simeq -\mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H} - \mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}k}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^4 \simeq \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H} + E_2 + \mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}k}f\tilde{F}\,\mathrm{e}^{-H}$,
$s_4^4 \simeq E_1 + \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}(1 + 2\,\mathrm{e}^{2\pi\mathrm{i}k}\cos 2\pi m)\tilde{F}\,\mathrm{e}^{-H}$.

R_5 solutions: $s_1^5 \simeq -\mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}$, $s_2^5 \simeq E_3 + \mathrm{i}\,\mathrm{e}^{3\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,
$s_3^5 \simeq -\mathrm{i}\,\mathrm{e}^{3\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$, $s_4^5 \simeq \mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H} + E_2$.

Sector 8. $\frac{5}{4}\pi < \arg\zeta < \frac{3}{2}\pi$

R_5 solutions as in sector 7, with dominancy reversed.

TABLE 2

Sectors 1 *and* 2: $-\frac{1}{2}\pi < \arg\zeta < 0$

$s(1) \simeq -\mathrm{i}f\tilde{F}\,\mathrm{e}^{-H}+E_0+g\tilde{G}\,\mathrm{e}^{H}$,

$s(2) \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(3) \simeq \mathrm{i}\,\mathrm{e}^{-3\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}-\mathrm{e}^{-2\pi\mathrm{i}k}E_{-1}-\mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$,

$s(4) \simeq E_0$.

Sectors 3 *and* 4: $0 < \arg\zeta < \frac{1}{2}\pi$,

$s(1) \simeq g\tilde{G}\,\mathrm{e}^{H}$,

$s(2) \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(3) \simeq -\mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}+E_1-\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(4) \simeq E_0$.

Sectors 5 *and* 6: $\frac{1}{2}\pi < \arg\zeta < \pi$

$s(1) \simeq g\tilde{G}\,\mathrm{e}^{H}$,

$s(2) \simeq \mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}+E_1-\mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}$,

$s(3) \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(4) \simeq E_0-2\cos 2\pi m\,\mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}$.

Sectors 7 *and* 8: $\pi < \arg\zeta < \frac{3}{2}\pi$

$s(1) \simeq g\tilde{G}\,\mathrm{e}^{H}+\mathrm{e}^{2\pi\mathrm{i}k}E_2+\mathrm{i}\,\mathrm{e}^{4\pi\mathrm{i}k}f\tilde{F}\,\mathrm{e}^{-H}$,

$s(2) \simeq -\mathrm{e}^{-\pi\mathrm{i}k}gh\tilde{G}\,\mathrm{e}^{H}-\mathrm{e}^{2\pi\mathrm{i}k}E_3-\mathrm{i}\,\mathrm{e}^{5\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(3) \simeq -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}k}\tilde{F}\,\mathrm{e}^{-H}$,

$s(4) \simeq -2\cos 2\pi m\,\mathrm{e}^{-2\pi\mathrm{i}k}g\tilde{G}\,\mathrm{e}^{H}+E_0-2\cos 2\pi m E_2$.

For definitions of sectors (1 to 8), and f, g, h, $\tilde{F}$, $\tilde{G}$, H, and E_r, see table 1.

We now apply our results to the physical problem. The interval $\mathscr{I}$ is now a segment, containing $\zeta = 0$, of the straight line formed by the rays $\arg\zeta\delta = 0$ and $\arg\zeta\delta = \pi$; see (2.21) and (2.24). If $\arg p = \pi$, these rays coincide with anti-Stokes lines, and our analysis gives no useful results. Since, however, this possibility corresponds to relatively uninteresting damped modes, we will consistently ignore it.

An inspection of table 2 shows that no solution exists which decays exponentially with increasing $|\zeta|$ both for $\arg\zeta\delta = 0$ and $\arg\zeta\delta = \pi$, i.e. there is no solution of physical interest that is localized within the critical layer. Next, we consider whether solutions exist which grow at most algebraically with increasing $|\zeta|$ for both $\arg\zeta\delta = 0$ and $\arg\zeta\delta = \pi$, i.e. whether solutions exist which match to ideal solutions on either side of the layer.

Consider the critical layer solutions

$$\left.\begin{aligned} I_1(\zeta) &= f(-k)\,\mathrm{e}^{-\pi\mathrm{i}k}s(1)+s(2)+s(3),\\ I_2(\zeta) &= -\frac{2\,\mathrm{e}^{-\pi\mathrm{i}k}\cos 2\pi m}{f(-k)}s(2)+s(4).\end{aligned}\right\} \tag{4.11}$$

These have the asymptotic properties

$$\left.\begin{aligned} &I_1(\zeta) \simeq E_1, \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \arg\zeta\delta = 0 \quad \text{or} \quad \pi,\\ &I_2(\zeta) \simeq E_0 \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \arg\zeta\delta = 0,\\ &I_2(\zeta) \simeq E_0-\frac{2\,\mathrm{e}^{-\pi\mathrm{i}k}\cos 2\pi m}{f(-k)}E_1, \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \arg\zeta\delta = \pi.\end{aligned}\right\} \tag{4.12}$$

In listing these results, exponentially small contributions, which change discontinuously across the Stokes lines $\arg\zeta = 0$ or π, have been omitted; i.e. the expansions (4.12) are asymptotic in the sense of Poincaré.

Suppose that the ideal equation (2.15) has been solved, in the case $p = 0$, in the two intervals $z_A < z < z_c$ to the 'left' and $z_c < z < z_B$ to the 'right' of the critical point z_c, and satisfy the required boundary conditions $\Psi(z_A) = \Psi(z_B) = 0$; see the discussion of § 2. In the neighbourhood of z_c, these solutions will assume the form

$$\left.\begin{aligned}\Psi &\simeq A_L(z_c-z)^{2m+\frac{1}{2}}+B_L(z_c-z)^{-2m+\frac{1}{2}} \quad (z\to z_c-0),\\ \Psi &\simeq A_R(z-z_c)^{2m+\frac{1}{2}}+B_R(z-z_c)^{-2m+\frac{1}{2}} \quad (z\to z_c+0),\end{aligned}\right\} \tag{4.13}$$

see (2.19). Although A_L/B_L and A_R/B_R are fixed by the boundary conditions, neither solution is, at this stage, determined to within an arbitrary multiplicative constant. The requirement that it should be possible to match (4.13) to a solution of the critical layer equation poses an eigenvalue problem for A, as we will now see.

In terms of the scaled variable ζ, (4.13) may be rewritten as

$$\Psi \simeq A_L\delta^{\frac{1}{2}+2m}\,e^{-\pi i(\frac{1}{2}+2m)}\,\zeta^{\frac{1}{2}+2m}+B_L\delta^{\frac{1}{2}-2m}\,e^{-\pi i(\frac{1}{2}-2m)}\,\zeta^{\frac{1}{2}-2m}, \quad \text{for} \quad |\zeta|\to\infty \quad \text{with} \quad \arg\zeta\delta=\pi,$$

$$\Psi \simeq A_R\delta^{\frac{1}{2}+2m}\zeta^{\frac{1}{2}+2m}+B_R\delta^{\frac{1}{2}-2m}\zeta^{\frac{1}{2}-2m}, \quad \text{for} \quad |\zeta|\to\infty \quad \text{with} \quad \arg\zeta\delta=0.$$

After multiplication by suitable constant factors, these solutions may, with the aid of the definitions of E_0 and E_1 (see table 1) and (4.12), be expressed as

$$\begin{aligned}\Psi = {}&[A_L\delta^{4m}\,e^{-4\pi im}[a_0(m)-Ja_1(m)]+B_L[a_0(-m)-Ja_1(-m)]]\,I_1\\ &-[A_L\delta^{4m}\,e^{-4\pi im}\,a_1(m)+B_La_1(-m)]\,I_2, \quad \text{if} \quad \arg\zeta\delta=\pi,\end{aligned} \tag{4.14}$$

$$\Psi = [A_R\delta^{4m}a_0(m)+B_Ra_0(-m)]\,I_1-[A_R\delta^{4m}a_1(m)+B_Ra_1(-m)]\,I_2, \quad \text{if} \quad \arg\zeta\delta=0, \tag{4.15}$$

where $(r = 1, 2)$

$$a_r(m) = \frac{2\pi i(\frac{1}{2}A+\frac{3}{4}+m-2k)\,e^{\pi irm}}{\Gamma(1+2m)\,\Gamma(\frac{3}{2}-2m)\,\Gamma(\frac{1}{2}-m-(-1)^rk)}, \qquad J = \frac{2\,e^{-\pi ik}\cos 2\pi m}{f(-k)}. \tag{4.16}$$

Equations (4.14) and (4.15) must represent the same solution of the critical layer equations, to within a multiplicative constant. It follows that

$$\begin{aligned}J[a_1(m)\,\delta^{2m}A_R+a_1(-m)\,\delta^{-2m}\,B_R]\times[a_1(m)\,\delta^{2m}A_L\,e^{-2\pi mi}+a_1(-m)\,\delta^{-2m}B_L\,e^{2\pi mi}]\\ =[a_0(m)\,a_1(-m)-a_1(m)\,a_0(-m)]\times[B_RA_L\,e^{-2\pi mi}-B_LA_R\,e^{2\pi mi}].\end{aligned} \tag{4.17}$$

This dispersion relation, which involves the ratios A_L/B_L and A_R/B_R (known from integrations from the boundaries), determines all possible eigenvalues, A. If m is purely imaginary, δ^{2m} and δ^{-2m} do not vanish in the limit $\delta\to 0$, but become increasingly oscillatory. Then A will itself change rapidly as $S\alpha\to\infty$, and will depend on A_L/B_L and A_R/B_R, i.e. it will be influenced by the location of the distant boundaries. Because of this, we may say that the diffusion layer surrounding $\zeta = 0$ is 'passive'. In contrast to this, if m is real, $\delta^{2m}\to 0$ and $\delta^{-2m}\to\infty$, as $\delta\to 0$. Then A is independent of A_L/B_L and A_R/B_R, i.e. it is not influenced by the distant boundaries. We may say now that the diffusion layer is 'active'. There are, in this case, two main possibilities, as (4.17) shows:

(i) neither B_L nor B_R vanishes (the general case). We must have

$$\text{either} \quad J = 0 \quad \text{or} \quad a_1(-m) = 0, \tag{4.18}$$

(ii) $B_L B_R = 0$ (the exceptional case) which may itself be divided into three subcases:

$$(a)\quad \frac{a_0(m)}{a_1(m)} - J = \frac{a_0(-m)}{a_1(-m)}, \quad \text{if} \quad B_L = 0 \quad \text{and} \quad B_R \neq 0, \tag{4.19}$$

$$(b)\quad \frac{a_0(-m)}{a_1(-m)} - J = \frac{a_0(m)}{a_1(m)}, \quad \text{if} \quad B_R = 0 \quad \text{and} \quad B_L \neq 0, \tag{4.20}$$

$$(c)\quad \text{either} \quad J = 0 \quad \text{or} \quad a_1(m) = 0, \quad \text{if} \quad B_L = B_R = 0. \tag{4.21}$$

Temporarily disregarding the $J = 0$ possibility, (4.18) requires that either

$$\tfrac{1}{2} + m + k = 1 - N, \tag{4.22}$$

where N is a positive integer, or

$$\tfrac{1}{2}A + \tfrac{3}{4} - m - 2k \equiv (2/A)\,(\tfrac{1}{4} + m)\,(\tfrac{1}{4} - m - \tfrac{1}{2}A) = 0. \tag{4.23}$$

Since $$(\tfrac{1}{2} + m + k) - 1 \equiv -(1/A)\,(\tfrac{1}{4} + m + \tfrac{1}{2}A)\,(\tfrac{1}{4} - m - \tfrac{1}{2}A),$$

condition (4.23) may be regarded as the missing $N = 0$ case of (4.22). The $J = 0$ possibility also leads to the same conditions, in the general case, although this requires a separate demonstration (§ 5) since the values of m concerned, are exceptional ($m = \tfrac{1}{4}(2r-1)$ where r is an integer). The second of the possibilities (4.21) gives the same results with m replaced by $-m$. Thus cases $B_L B_R \neq 0$ (upper sign) and $B_L = B_R = 0$ (lower sign) require

$$\tfrac{1}{2} \pm m + k = 1 - N \quad (N = \text{non-negative integer}). \tag{4.24}$$

Conditions (4.19) and (4.20) lead to demands on m which cannot be met.

From (3.13) and Kummer's first theorem, we have

$$q(m,s) = \frac{1}{\Gamma(1+2m)}\, e^{\frac{1}{2}s^2} s^{2m-\frac{3}{2}} [(\tfrac{1}{2} + m + k)\, {}_1F_1(-\tfrac{1}{2} + m + k; 1 + 2m; -s^2)$$
$$+ (\tfrac{1}{2}A + \tfrac{1}{4} - k)\, {}_1F_1(\tfrac{1}{2} + m + k; 1 + 2m; -s^2)],$$

and hence the solutions given by (4.24) are characterized by the fact that one or other of the $q(\pm m, s)$ has terminating ${}_1F_1$ series. These are the solutions obtained by J.G.C. (§IV; cf. (50) noting that Fourier transforms are used so that their $\alpha < 0$). Equation (4.24) determines the eigenvalues A for given m. Using the definition (2.29) of k, we obtain

$$A^2 + 4A(\pm m + N) + 4m^2 - \tfrac{1}{4} = 0, \tag{4.25}$$

which is condition (59) of J.G.C. Its solution is

$$A = -2(N \pm m) \pm \sqrt{\{4(N \pm m)^2 - 4m^2 + \tfrac{1}{4}\}}. \tag{4.26}$$

For $0 < m < \tfrac{1}{4}$, A is real and, in general situations (upper sign with m), it is positive. Equation (2.24) then gives positive values for p, and hence instabilities. For $m > \tfrac{1}{4}$, A is negative in general situations, giving damped oscillations.

5. Exceptional cases

The general results of § 4 become invalid for the exceptional values of m which make $1/\Gamma(1-2m)$ or $1/\Gamma(\tfrac{3}{2}-2m)$ zero, and special circumstances also relate to the vanishing of $\tfrac{1}{2}A + \tfrac{3}{4} + m - 2k$:

(i) If $m = \tfrac{1}{2}n$ where n is a positive integer, the inner saddle point makes only one independent contribution, and those from the outer saddle points vanish, since each contains a factor $\sin 2\pi m$. For $m = 0$, the solution (3.12) for $Q(s)$ is degenerate. These situations arise from the fact that one of the solutions (3.8) of Whittaker's equation (3.5) becomes invalid, and the usual limiting argument must be used to find another. No physical interest attaches to this mathematical point, and the case will from now onwards be ignored.

(ii) If $2m = n+\frac{3}{2}$ where n is a non-negative integer, the inner saddle point again yields only one independent contribution, and the solutions of table 2 are no longer linearly independent, though the analysis remains valid if the E_r are interpreted as in (5.1) below. In such cases the origin is no longer a branch point, and the $q(m, s)$ defined by (3.13) is regular at the origin. It is in fact a zero of multiplicity n, whereas $q(-m, s)$ has a pole of order $-n-3$. Hence the only solution associated with the inner saddle point is a polynomial of degree $n+2$ (see, for instance, (3.1) with $\mathscr{C}$ chosen as the appropriate C_N, or simply a contour encircling the origin), and is valid for all ζ. This is also shown by solution $s(4)$ when $\cos 2\pi m$ is set zero, as long as the following interpretation is used for E_r:

$$E_r = -\frac{2\pi \mathrm{i}(\frac{1}{2}A+\frac{3}{4}-m-2k)}{\Gamma(1-2m)\,\Gamma(\frac{1}{2}+m-(-1)^r k)}\,\mathrm{e}^{-\pi \mathrm{i} r m}\,\frac{\zeta^{n+2}}{(n+2)!}. \tag{5.1}$$

The further solution, required to complete the set of four, may be found by choosing $Q(s)$ and $K(s)$ as the $m > 0$ cases of (3.10) and (3.11), and the contour $\mathscr{C}$ of solutions (3.1) as an appropriate ray R_N. The inner saddle point contribution to this new solution for large $|\zeta|$ may be found as in § 4. The leading order integral to be considered is

$$\int_{\mathscr{C}_{s_0}} \mathrm{e}^{s\zeta} s^n\,\mathrm{d}s,$$

where $\mathscr{C}_{s_0}$ is a contour running along the ray $\arg s = \pi - \arg\zeta$ from the origin to infinity. Transforming to the t plane as before, the integral becomes, with exponentially small error,

$$\left(-\frac{1}{\zeta}\right)^{n+1}\int_0^\infty \mathrm{e}^{-t}\,t^n\,\mathrm{d}t = n!\left(-\frac{1}{\zeta}\right)^{n+1}. \tag{5.2}$$

A convenient set of four solutions associated with the ray R_N is now

$$\boldsymbol{s}^N = \begin{bmatrix} \Psi_{N+1}(\zeta;\gamma_N^+) \\ \Psi_{N-2}(\zeta;\gamma_N^-) \\ \Psi^0(\zeta) \\ \Psi(\zeta;R_N) \end{bmatrix}, \tag{5.3}$$

where s_1^N and s_2^N are defined as in (3.25), while

$$\Psi^0(\zeta) = \int_{C_N} \mathrm{e}^{s\zeta}\,q(-m,s)\,\mathrm{d}s \simeq \frac{2\pi\mathrm{i}(\frac{1}{2}A+\frac{3}{4}-m-2k)}{\Gamma(1-2m)\,(n+2)!}\,\zeta^{n+2}, \tag{5.4}$$

and

$$\Psi(\zeta;R_N) = \int_{R_N} \mathrm{e}^{s\zeta}\,q(m,s)\,\mathrm{d}s. \tag{5.5}$$

Proceeding as in § 4, the relation

$$\boldsymbol{s}^N = T(k(-1)^N, N)\,\boldsymbol{s}^{N+1}, \tag{5.6}$$

may be established where

$$T(k,N) = \begin{bmatrix} 0 & \mathrm{e}^{-2\pi\mathrm{i}k} & 0 & 0 \\ -1 & -f(k)\,\mathrm{e}^{-\pi\mathrm{i}k} & -\dfrac{\mathrm{e}^{-\pi\mathrm{i}m(N-2)}}{\Gamma(\frac{1}{2}+m-k)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dfrac{\pi\mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}(k+mN)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)} & 0 & 1 \end{bmatrix}. \tag{5.7}$$

As a check, repeated application of (5.6) yields, after some reductions using $\cos 2\pi m = 0$,

$$s^N = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s^{N+4}. \tag{5.8}$$

As far as s_1^N and s_2^N are concerned, these results agree with (3.26). The s_3^N and s_4^N solutions do not partake in the sign change, as is immediately clear from (5.4) and (5.5), recalling that $2m-\frac{3}{2}=n$. Again proceeding as in § 4, the new solution s_4^N may be evaluated for large $|\zeta|$, and used to supplement table 1, which is still valid. The results are as follows:

	$s_4^2 \simeq X$	for $-\frac{1}{2}\pi < \arg\zeta < -\frac{1}{4}\pi$,
$s_4^2 \simeq X$	$s_4^3 \simeq X + \dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}m}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\,\mathrm{e}^{-H}$	for $-\frac{1}{4}\pi < \arg\zeta < 0$,
$s_4^2 \simeq X - \dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}m}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\,\mathrm{e}^{-H}$	$s_4^3 \simeq X$	for $0 < \arg\zeta < \frac{1}{4}\pi$,
$s_4^3 \simeq X$	$s_4^4 \simeq X - \dfrac{\mathrm{i}\pi\,\mathrm{e}^{-\pi\mathrm{i}(3m+k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m+k)}g\tilde{G}\,\mathrm{e}^{H}$	for $\frac{1}{4}\pi < \arg\zeta < \frac{1}{2}\pi$,
$s_4^3 \simeq X + \dfrac{\mathrm{i}\pi\,\mathrm{e}^{-\pi\mathrm{i}(3m+k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m+k)}g\tilde{G}\,\mathrm{e}^{H}$	$s_4^4 \simeq X$	for $\frac{1}{2}\pi < \arg\zeta < \frac{3}{4}\pi$,
$s_4^4 \simeq X$	$s_4^5 \simeq X - \dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}(2m-k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\,\mathrm{e}^{-H}$	for $\frac{3}{4}\pi < \arg\zeta < \pi$,
$s_4^4 \simeq X + \dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}(2m-k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\,\mathrm{e}^{-H}$	$s_4^5 \simeq X$	for $\pi < \arg\zeta < \frac{5}{4}\pi$,
$s_4^5 \simeq X$		for $\frac{5}{4}\pi < \arg\zeta < \frac{3}{2}\pi$,

where
$$X = \frac{\frac{1}{2}A+\frac{3}{4}+m-2k}{\Gamma(1+2m)}\,n!\left(-\frac{1}{\zeta}\right)^{n+1}. \tag{5.9}$$

From the supplemented table 1, the following solution $s(5)$ may now be found to supplement table 2:

$$s(5) \simeq \begin{cases} \dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}m}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\mathrm{e}^{-H}+X & \text{for } -\frac{1}{2}\pi < \arg\zeta < 0, \\ X & \text{for } 0 < \arg\zeta < \frac{1}{2}\pi, \\ X+\dfrac{\mathrm{i}\pi\,\mathrm{e}^{-\pi\mathrm{i}(3m+k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m+k)}g\tilde{G}\,\mathrm{e}^{H} & \text{for } \frac{1}{2}\pi < \arg\zeta < \pi, \\ \dfrac{\mathrm{i}\pi\,\mathrm{e}^{-\pi\mathrm{i}(3m+k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m+k)}g\tilde{G}\,\mathrm{e}^{H}+X+\dfrac{\mathrm{i}\pi\,\mathrm{e}^{-\pi\mathrm{i}(3m-k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m+k)}E_2 & \\ \qquad -\dfrac{\pi\,\mathrm{e}^{-2\pi\mathrm{i}(m-2k)}}{\sin 2\pi m\,\Gamma(\frac{1}{2}+m-k)}\tilde{F}\mathrm{e}^{-H} & \text{for } \pi < \arg\zeta < \frac{3}{2}\pi. \end{cases}$$

The argument dealing with the continuation of the ideal solution through the critical layer may now be extended to include these exceptional cases. In general situations, i.e. $B_\mathrm{L} B_\mathrm{R} \neq 0$ in the notation of § 4, $s(5)$ provides the relevant solution through the layer, if and only if the condition (4.22) is satisfied for non-negative N. An inspection of equation (4.26) shows that since $m > \frac{1}{4}$, such modes are damped. In exceptional circumstances, however, when $B_\mathrm{L} = B_\mathrm{R} = 0$, solution $s(4)$ apparently provides the relevant solution through the layer for any A. There is, in fact, no critical layer in this case, the ideal solution with leading term $(z-z_c)^{n+2}$ near $z = z_c$, providing the solution throughout $\mathscr{I}$.

(iii) From equation (4.23), the expression $\frac{1}{2}A+\frac{3}{4}+m-2k$ is zero when $m=\frac{1}{4}$, and once again the inner saddle point makes only one independent contribution. In this case, however, conditions (3.16) are not satisfied at the origin by any of the possible $Q(s)$ and $K(s)$, so that the extra solution may not be provided by a contour terminating at the origin. As in the last case, the contribution from a relevant contour C_N, or a closed contour encircling $s = 0$, is a polynomial and is valid for all ζ. The analysis of § 4 leading to tables 1 and 2 is again correct, but provides only three linearly independent solutions.

Since (2.11) and (2.20) show that Ψ is an integral function of m, the fourth solution may be obtained by a limiting argument as follows: consider, for general m, the solution

$$s(m,\zeta) = f(-k)\,\mathrm{e}^{-\pi i k}\,s(1)+s(2)+s(3)-\mathrm{e}^{-\pi i m}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(\frac{1}{2}+m+k)}\,s(4). \tag{5.10}$$

The exponential terms contain the factor $\cos 2\pi m$, and the ideal terms contain the expression $\frac{1}{2}A+\frac{3}{4}+m-2k$ or $\cos 2\pi m$, thus $s(\frac{1}{4},\zeta)\equiv 0$. The expression

$$\bar{s}(\zeta) = \lim_{m\to\frac{1}{4}} \frac{s(m,\zeta)}{m-\frac{1}{4}} \tag{5.11}$$

does not, however, vanish identically. It provides the required fourth solutions for $m=\frac{1}{4}$. On division by the constant $\sqrt{2}\,\mathrm{i}(A-1)\,\Gamma(\frac{1}{4}-\frac{1}{4}A)\exp(-\frac{1}{4}\pi \mathrm{i}A)/(\sqrt{\pi}A)$, it is found that

$$\bar{s}(\zeta) \simeq 2\sqrt{2} \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad 0<\arg\zeta<\tfrac{1}{2}\pi,$$

$$\bar{s}(\zeta) \simeq 2\sqrt{2}-\frac{\mathrm{i}\pi^{\frac{3}{2}}A\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}(1+A)}}{\sqrt{2}\,\Gamma(\frac{5}{4}+\frac{1}{4}A)}\tilde{G}\,\mathrm{e}^{H} \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \tfrac{1}{2}\pi<\arg\zeta<\pi.$$

Now let

$$s(6) = \bar{s}(\zeta)+\frac{\sqrt{(2\pi)}\,A\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{A-1}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{4}A)\,s(2), \tag{5.12}$$

then, to leading order

$$s(6) \simeq 2\sqrt{2} \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \arg\zeta\delta = 0,$$

$$s(6) \simeq 2\sqrt{2}+\pi\sqrt{2}\frac{A}{A-1}\frac{\Gamma(\frac{3}{4}+\frac{1}{4}A)}{\Gamma(\frac{5}{4}+\frac{1}{4}A)}\zeta \quad \text{for} \quad |\zeta|\to\infty \quad \text{and} \quad \arg\zeta\delta=\pi.$$

The solutions $s(2)$, $s(3)$, $s(4)$ and $s(6)$ provide a linearly independent set when $m=\frac{1}{4}$. This case, relevant to the tearing mode, has been considered in detail by Gibson & Kent (1971), and our solutions agree with theirs, for $\arg\zeta\delta = 0$ and π, to leading order (observe that our A is their μ^2). It may be seen that, on multiplication by a factor of $\sqrt{(\frac{1}{2}\pi)}\,\mathrm{i}\,\mathrm{e}^{-\pi i k}$, our solution $s(2)$ can be identified as their solutions (6.27) and $(6.27)_1$; also, on multiplication by $\sqrt{(\frac{1}{2}\pi)}\,\mathrm{e}^{\pi i k}$, our $s(3)$ leads to their (6.26) and $(6.26)_1$. Our $s(4)$, which is a multiple of ζ, corresponds to their (6.3), and finally our $s(6)$ corresponds to their (6.31) and (6.32). The coefficient of ζ in $s(6)$ for $\arg\zeta=\pi$ becomes, when approximated for small A,

$$-4\sqrt{2}\,\pi A\,\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{1}{4}), \tag{5.13}$$

cf. their (6.32) and (6.36). The continuation of the ideal solution through the singular layer in this case is given fully by Gibson & Kent.

6. The localized gravitational modes

It is now demonstrated how perturbations of short wavelength $(1/\alpha)$ can give rise to instabilities which are localized inside the critical layer. The disturbances are governed by the critical layer equation (2.22) with appropriate choices of layer width $\delta\,[< O(1)]$ and eigenvalue $p\,[< O(1)]$, for the postulated $\alpha\,[> O(1)]$. There are three main cases to consider corresponding to three different orderings of α, namely (i) $\alpha > O(S^{\frac{1}{2}})$, (ii) $\alpha = O(S^{\frac{1}{2}})$ and (iii) $O(1) < \alpha < O(S^{\frac{1}{2}})$. In case (iii), the solution exhibits a double structure, one ('inner') layer lying within the other ('outer'). The outer solution will be denoted by ψ_0, and the corresponding stretched coordinate by η. (The use of η for magnetic diffusivity will henceforth be abandoned.)

Case (i): $\alpha > O(S^{\frac{1}{2}})$

The width of the critical layer is here $O(\alpha^{-\frac{1}{2}}S^{-\frac{1}{4}})$ and it is found that $p = \alpha^{-1}G^{\frac{1}{2}}[1+O(S^{\frac{1}{2}}/\alpha)]$. In fact the choice

$$\delta = G^{\frac{1}{8}}(F_c')^{-\frac{1}{2}}S^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}, \quad p = \alpha^{-1}G^{\frac{1}{2}}(1-\tfrac{1}{2}\lambda G^{-\frac{1}{4}}F_c'S^{\frac{1}{2}}\alpha^{-1}), \tag{6.1}$$

where $\arg\delta$ is zero for $F_c' > 0$ and $\frac{1}{2}\pi$ for $F_c' < 0$, gives, in leading order, the equation

$$\left(\frac{d^2}{d\zeta^2}-\zeta^2+\lambda\right)\frac{\psi}{\zeta} = 0. \tag{6.2}$$

This has the general solution

$$\psi = A\zeta^{\frac{1}{2}}W_{\frac{1}{4}\lambda,\frac{1}{4}}(\zeta^2)+B\zeta^{\frac{1}{2}}W_{-\frac{1}{4}\lambda,\frac{1}{4}}(-\zeta^2), \tag{6.3}$$

where A and B are constants and $W_{k,m}$ is the Whittaker function. Suppose first that $\arg\delta = 0$. In order that the solution (6.3) is 'localized to the right' [i.e. in order that $\psi \to 0$ as $\zeta \to \infty\exp(0)$], it is clearly necessary that $B = 0$; cf., for example, Slater (1960, § 4.1.3). From the requirement that it should also be 'localized to the left' [i.e. that $\psi \to 0$ as $\zeta \to \infty\,e^{i\pi}$] it then follows that

$$\frac{1}{\Gamma(\frac{1}{2}(1-\lambda))} = 0. \tag{6.4}$$

It is now seen that an infinite set of eigenvalues p exist for which the λ defined in (6.1) are given by

$$\lambda = 1+2r, \tag{6.5}$$

where r is a non-negative integer; the corresponding eigenfunctions are

$$\psi = \zeta^{\frac{1}{2}}W_{\frac{1}{4}(1+2r),\frac{1}{4}}(\zeta^2). \tag{6.6}$$

If $\arg\delta = \frac{1}{2}\pi$, a similar analysis leads to (6.5) and (6.6) with λ replaced by $-\lambda$, and ζ^2 by $-\zeta^2$.

Case (ii): *the fast interchange modes* (*J.G.C.*); $\alpha = O(S^{\frac{1}{2}})$

The width of the critical layer is here $\delta = O(S^{-\frac{1}{2}})$ and it is found that $p = O(S^{-\frac{1}{2}})$. All terms in (2.22) contribute to leading order, and the fast interchange mode (cf. J.G.C. § III) is recovered. More specifically the choice

$$\alpha = \left(\frac{E^3F_c'^2}{A}\right)^{\frac{1}{4}}S^{\frac{1}{2}}, \quad \delta = \left(\frac{A}{EF_c'^2}\right)^{\frac{1}{4}}S^{-\frac{1}{2}}, \quad \text{and} \quad p = \left(\frac{A^3F_c'^2}{E}\right)^{\frac{1}{4}}S^{-\frac{1}{2}}, \tag{6.7}$$

together with the introduction of new variables Ψ and V, defined by

$$\psi = S^{-\frac{3}{2}}\Psi \quad \text{and} \quad W = (E^3AF_c'^2)^{\frac{1}{4}}V, \tag{6.8}$$

into equations (2.9) and (2.10) leads to

$$\mathrm{d}^2\Psi/\mathrm{d}\zeta^2 = (A+E)\,\Psi + A\zeta V \tag{6.9}$$

and

$$A\,\mathrm{d}^2V/\mathrm{d}\zeta^2 = (AE+4m^2-\tfrac{1}{4}+A\zeta^2)\,V + A\zeta\Psi. \tag{6.10}$$

Equations (6.9) and (6.10) reduce to (2.25) and (2.26) when $E = 0$. Unfortunately we have been unable to determine the structure of the solutions of (6.9) and (6.10) through the critical layer, and thus to provide a more general analysis that would have included that of §§ 3 and 4. Apart from the case $E = 0$, the equations have been solved for $E \gg A$ by J.G.C., who also obtained some numerical solutions. The analytic treatment of the equations is the remaining outstanding problem connected with the gravitational instabilities in the sheet pinch. Its importance should, however, not be overstressed; see remark after case (iii) below.

Case (iii): $O(1) < \alpha < O(S^{\frac{1}{2}})$

The width of the inner layer is here $O(S\alpha)^{-\frac{1}{3}}$, while that of the outer layer is $O(1/\alpha)$. It is found that $p = O(S\alpha)^{-\frac{1}{3}}$. With $\delta = 1/\alpha$, the outer equation is, to leading order,

$$\frac{\mathrm{d}^2\psi_0}{\mathrm{d}\eta^2} + \left(-1+\frac{\frac{1}{4}-m^2}{\eta^2}\right)\psi_0 = 0, \tag{6.11}$$

of which the general solution is

$$\psi_0 = AM_{0,2m}(2\eta) + BM_{0,-2m}(2\eta), \tag{6.12}$$

where A and B are constant, and $M_{k,m}$ denotes the Kummer function (cf. (3.5) and (3.8) above). Appropriate values of A/B can be found both on the left ($\arg\eta = \pi$) and right ($\arg\eta = 0$) of the critical point such that the solution is exponentially small for $|\eta| \to \infty$ in both cases. For $\eta \to 0$, these solutions will take the form

$$\left.\begin{aligned} \psi_0 &\simeq A_{\mathrm{L}}(-\eta)^{\frac{1}{2}+2m} + B_{\mathrm{L}}(-\eta)^{\frac{1}{2}-2m} \quad (\eta \to 0-),\\ \psi_0 &\simeq A_{\mathrm{R}}\eta^{\frac{1}{2}+2m} + B_{\mathrm{R}}\eta^{\frac{1}{2}-2m} \quad (\eta \to 0+), \end{aligned}\right\} \tag{6.13}$$

(cf. (4.13)). Thus, since the inner layer scaling is that defined by (2.23), the matching problem of § 4 is recovered, and the conclusions of that section follow.

In all three cases positive eigenvalues p exist giving instabilities. The order of magnitude of p decreases continuously as α increases. In physical units, the growth rate αp of the disturbance increases continously from $O(\alpha^{\frac{2}{3}}S^{-\frac{1}{3}}\tau_a^{-1})$ for the slow interchange mode to $O(\tau_a^{-1})$ for $\alpha \geqslant O(S^{\frac{1}{2}})$. Although the latter instabilities have the same growth rate as that arising in the ideal theory, they are not believed to be of great practical import, since they are easily stabilized (see, for example, Stringer 1967; see also F.K.R. § VII, and J.G.C. § VI). The outer layer of thickness $1/\alpha$ which exists when $O(1) < \alpha < O(S^{\frac{1}{2}})$ gradually thickens to become the mainstream of the slow interchange mode for $\alpha \leqslant O(1)$.

7. The oscillatory modes

The oscillatory resistive modes arising when F has an extremum within $\mathscr{I}$, at which $F \neq 0$, will now be considered (see § 2). Resistive effects become important in a critical layer where F attains its extreme value, the governing equation being (2.32). This equation may be solved by the Laplace integral

$$\psi(\zeta) = \int_{\mathscr{C}} \mathrm{e}^{s\zeta}\, Q(s)\,\mathrm{d}s, \tag{7.1}$$

where $Q(s)$ is a solution of the equation

$$\frac{1}{4s^2}\frac{\mathrm{d}^2Q}{\mathrm{d}s^2}+\frac{1}{2s^3}\frac{\mathrm{d}Q}{\mathrm{d}s}+\left(-\frac{1}{4}+\frac{A}{4s^2}+\frac{B}{4s^4}\right)Q=0, \tag{7.2}$$

and $\mathscr{C}$ is a contour in the complex s plane, chosen so that

$$\left[\mathrm{e}^{s\zeta}\left\{(\zeta s^2+4s)\,Q+s^2\frac{\mathrm{d}Q}{\mathrm{d}s}\right\}\right]_{\mathscr{C}}=0. \tag{7.3}$$

The change of variables
$$u=s^2,\quad Q=u^{-\frac{3}{4}}M, \tag{7.4}$$

in (7.2) then leads to
$$\frac{\mathrm{d}^2M}{\mathrm{d}u^2}+\left(-\frac{1}{4}+\frac{k}{u}+\frac{\frac{1}{4}-m^2}{u^2}\right)M=0, \tag{7.5}$$

where
$$k=\tfrac{1}{4}A\quad\text{and}\quad \tfrac{1}{4}-m^2=\tfrac{1}{4}B+\tfrac{3}{16}. \tag{7.6}$$

Hence the general solution for $Q(s)$ is any linear combination of the solutions $q(\pm m,s)$, where

$$q(m,s)=\frac{1}{\Gamma(1+2m)}\,s^{-\frac{1}{2}+2m}\,\mathrm{e}^{-\frac{1}{2}s^2}\,{}_1F_1(\tfrac{1}{2}+m-k;\,1+2m;\,s^2) \tag{7.7}$$

(cf. (3.5) *et. seq.*).

The solutions (7.1) may thus be investigated for large $|\zeta|$ as in §§3 to 5, the solutions $Q_r(s)$ of (3.12) occupying a central place in the discussion, but with $q(m,s)$ defined as in (7.7). The contours defined in §3 may be used without change, the appropriately modified asymptotic expansions of (3.20) and (3.21) leading to exactly the same linking matrix T. The asymptotic analysis of §4 needs minor modification, and tables 1 and 2 remain valid with the interpretation

$$\left.\begin{aligned}
&f=f(k)=\frac{2\pi}{\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)},\quad g=1,\quad h=f(-k),\\
&\tilde{F}=-\sqrt{\frac{2}{\pi}}\sin 2\pi m\,\zeta^{-\frac{3}{2}-2k},\quad \tilde{G}=\sqrt{\frac{2}{\pi}}\sin 2\pi m\,\zeta^{-\frac{3}{2}+2k},\quad H=\tfrac{1}{2}\zeta^2,\\
&E_r=L_r(m,\zeta)-L_r(-m,\zeta),\\
\text{where}\quad &L_r(m,\zeta)=\frac{2\pi\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}rm}}{\Gamma(1+2m)\,\Gamma(\frac{1}{2}-2m)\,\Gamma(\frac{1}{2}-m-(-1)^r k)}\left(\frac{1}{\zeta}\right)^{\frac{1}{2}+2m}.
\end{aligned}\right\} \tag{7.8}$$

From (2.31), if the upper sign is selected, $\arg\delta=\pm\frac{1}{8}\pi$ according as F_c'' is positive or negative, so that $z>z_c$ corresponds to $\arg\zeta=\mp\frac{1}{8}\pi$, and $z<z_c$ corresponds to $\arg\zeta=\pi\mp\frac{1}{8}\pi$. The signs of $\arg\delta$ and therefore of $\arg\zeta$ are reversed if the lower sign in (2.31) is selected. The change in the ζ direction does not take the matching discussion out of the sectors used in §4, so that the solutions $I_1(\zeta)$ and $I_2(\zeta)$ of (4.11) and (4.12) may again be used. The ideal solutions of (4.13) remain valid if the exponents $\frac{1}{2}\pm 2m$ are changed to $-\frac{1}{2}\pm 2m$. The matching then continues as before with the interpretation

$$a_r(m)=\frac{\mathrm{e}^{\pi\mathrm{i}rm}}{\Gamma(1+2m)\,\Gamma(\frac{1}{2}-2m)\,\Gamma(\frac{1}{2}-m-(-1)^r k)}, \tag{7.9}$$

J remaining unchanged, and leads to a matching condition similar to (4.17). For 'active' layers the cases (4.19) and (4.20) again give no solutions, but (4.18) and (4.21) lead to

$$\tfrac{1}{2}\pm m+k=-N, \tag{7.10}$$

for $m > 0$, where N is a non-negative integer, the upper sign referring to general situations ($B_L \neq 0$, $B_R \neq 0$). Equations (7.6) then give

$$A = -4N - 2 \mp (1-4B)^{\frac{1}{2}}, \tag{7.11}$$

for $B < \frac{1}{4}$. Hence, if F^2 has a maximum ($F_c F_c'' < 0$), the critical layer is 'active' if $G > \frac{1}{4}F_c F_c''$, and eigensolutions exist, the second of equations (2.31) showing that they are damped, the rate of damping increasing with mode number N. Similarly, if F^2 has a minimum, the critical layer is 'active' if $G < \frac{1}{4}F_c F_c''$, all eigensolutions again being damped, the rate of damping increasing with N. In other words, if G is sufficiently large there is an infinity of damped resistive modes which approach $\pm \mathrm{i}F_{\max}$ in the p plane as $S\alpha \to \infty$, and similarly if G is sufficiently small there is another infinity of damped resistive modes which approach $\pm \mathrm{i}F_{\min}$.

The exceptional cases (i) and (ii) of § 5 are also relevant to the discussion. The comments on case (i) stand without alteration. In case (ii) we consider $m = n + \frac{1}{2}$, where n is a non-negative integer. This now gives $q(-m, s)$ a pole of order $-n-1$, leading to a polynomial solution of degree n. Table 2 remains valid with the final definition of (7.8) replaced by

$$E_r = -\frac{2\pi \mathrm{i}\, \mathrm{e}^{-\pi \mathrm{i} r m}}{\Gamma(1-2m)\,\Gamma(\frac{1}{2}+m-(-1)^r k)} \frac{\zeta^n}{n!}. \tag{7.12}$$

A modified set of four solutions may be selected as in § 5 ,where (5.4) now becomes

$$\Psi^0(\zeta) = \int_{C_N} \mathrm{e}^{s\zeta} q(m,s)\, \mathrm{d}s \simeq \frac{2\pi \mathrm{i}}{\Gamma(1-2m)} \frac{\zeta^n}{n!}. \tag{7.13}$$

This leads to the same modified matrix T of (5.7), and the solutions s_4^N, and $s(5)$ are also valid with the interpretation

$$X = \frac{n!}{\Gamma(1+2m)} \left(-\frac{1}{\zeta}\right)^{n+1}. \tag{7.14}$$

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